

Theory vs Practice in the Teaching of Mathematics

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The Teaching of Mathematics

- Mathematics has been taught for the past 4000 years
- Much of the teaching is through problem solving
- Although the problems often appear to be practical, in very many cases they are only made-up problems that a practitioner would never have to solve
- Problems are taught so that students will learn techniques of problem solving
- There is theory behind the procedures for solving problems, and it is important to learn the theory to develop flexibility in solving problems, but it is not always taught.

Mesopotamia, c. 1800 BCE

Typical Mesopotamian problems:

The field and 2 equal-sides added together gives 120.

The field and equal-sides are what? [$x^2 + 2x = 120$]

Find the length of a reed if it is applied an unknown number of times to measure a length of 4 cubits, where for each application of the reed, a piece of length 1 finger falls off. (30 fingers = 1 cubit)

$$[n(n + 1)/2 = 120]$$

Why did Mesopotamian scribes learn to solve quadratic equations?

- No real world problems that actually required quadratic equations
- Tablets often have lists of problems of increasing difficulty, so they teach problem solving skills:
 - Determine what procedure will work with a given problem
 - Solve simple problems with a basic algorithm
 - Reduce complicated problems to ones already solved
 - Learn when to modify algorithm
 - Recognize when unusual-seeming problems are just restatements of standard problems



Rhind Papyrus (c. 1800 BCE)

- Accurate reckoning. The entrance into the knowledge of all existing things and all obscure secrets.
- To develop this knowledge, one learns to solve problems
- In this case, most of problems are closely related to problems that an official needs to deal with
- If it is said to thee, divide 10 hekats of barley among 10 men so that the difference of each man and his neighbor in hekats of barley is $\frac{1}{8}$, what is each man's share?



Mathematical Teaching in Greece

The mathematical part of the curriculum for future leaders of the state had five parts:

- Arithmetic (theory)
- Plane geometry
- Solid geometry
- Astronomy (theoretical – study of solid bodies in circular motion)
- Harmonics (theoretical musical studies)

(From Plato's *Republic*)



Mathematics in Greece

- Why should theoretical mathematics be taught to future leaders of the society?
- So they would have “higher-level cognitive performance” and be able to perform their duties as leaders
- Interestingly, the Greek method of teaching theoretical mathematics, beginning with definitions and axioms and using logic to prove new theorems, became the model for Western mathematics.



Issues in Greek mathematics

- Unfortunately, most Greek theoretical mathematics texts do not explain how the authors discovered the theorems. And mathematics is certainly about more than just “proving theorems.”
- Virtually the only work that allows us to see how theorems were discovered is the Method of Archimedes – and even that was at a very high level
- Of course, there were merchants and artisans in Greece who needed and used mathematics, but the details of their mathematics have not reached us



Mathematics in Islam

- Greek influence was very strong, so many Islamic mathematics wrote following the Greek models
- Yet there was also influence from ancient Mesopotamia and from India
- Algebra develops in Islam based on Mesopotamian and possibly Indian models, and so is based on solving problems
- Basic algebra algorithms are proved using geometry – sometimes Euclidean but also via Mesopotamian methods - but are then applied to solve problems

Al-Khwarizmi – c. 825 CE

Earliest algebra text

- That fondness for science, by which God has distinguished the Imam al-Ma'mun, the Commander of the Faithful...that affability and condescension which he shows to the learned,... has encouraged me to compose a short work on calculating by *al-jabr* and *al-muqabala*, confining it to what is easiest and most useful in arithmetic, such as men constantly require in cases of inheritance, legacies, partition, law-suits, and trade, and in all their dealing with one another, or where the measuring of lands, the digging of canals, geometrical computation, and other objects of various sorts and kinds are concerned.



Problem from al-Khwarizmi (1)

- Although it appears that the book would be concerned with practical problems, most of the examples are not practical.
- I have divided ten into two parts, and having multiplied each part by itself, I have put them together, and have added to them the difference of the two parts previously to their multiplication, and the amount of all this is fifty-four.
- $(10 - x)^2 + x^2 + (10 - x) - x = 54$ $x^2 + 28 = 11x$

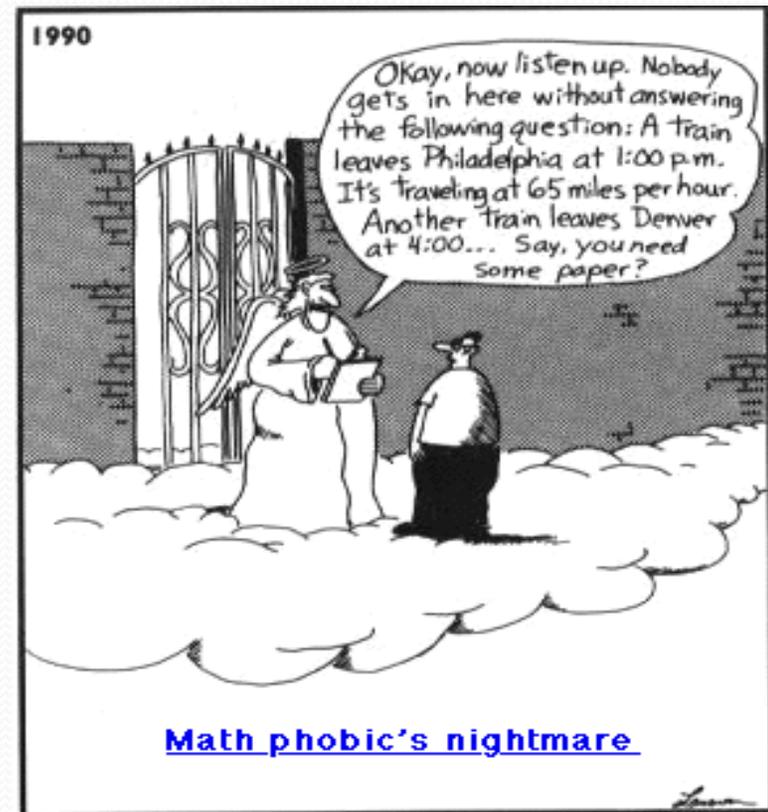
Problem from al-Khwarizmi (2)

- Here is an example of a quasi-real problem:
- You divide one dirhem among a certain number of men. Now you add one man more to them, and divide again one dirhem among them. The quota of each is then one-sixth of a dirhem less than at the first time.
- $[1/x - 1/(x + 1) = 1/6]$

Abu Bakr al-Karaji (11th c.)

- An early travel problem:
- Of two travelers going in the same direction, the first goes 11 miles per day, while the second, leaving five days later, goes successively each day 1 mile, 2 miles, 3 miles, and so on. In how many days will the second traveler overtake the first?
[$\frac{1}{2}(x^2+x)=11x+55$]

Answer: $\sqrt{(220\frac{1}{4})} + 10\frac{1}{2} \approx 25.3$ days



The Far Side by Gary Larson.
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Practical Problems in Islam

- Many Islamic texts contained very practical problems, dealing frequently with trade and commerce. The basic technique presented was the “rule of three”, that is, solving the proportion problem $a:x = b:c$.
- Al-Khwarizmi even had many problems dealing with inheritance: A woman dies, leaving her husband, son and three daughters, but she also leaves to a stranger $\frac{1}{8} + \frac{1}{7}$ of her estate. Calculate the shares of each.
- This can be solved, once one knows Islamic inheritance laws, by using a linear equation.



Trigonometry in Islam

- Trigonometry is a very practical subject and was developed intensely in Islam:
- Abu l-Rāyhan al-Bīrūnī (973-1055) – *Book of the Determination of Coordinates of Localities*
 - Determine circumference of the earth
 - Determine longitude and latitude of localities
 - Determine distance between localities
- al-Bīrūnī also solved the problem of the *qibla*, determining the direction of prayer.
- But all of these required a certain amount of theory.



Trigonometry in Islam

- Nasīr al-Dīn al-Tūsī (1201-1274), *Treatise on the Complete Quadrilateral*
- Presented a complete theoretical account of plane and spherical trigonometry
- Culminated with methods for finding unknown sides and angles in spherical triangles, given that certain sides and angles were known.

Leonardo of Pisa (Fibonacci)

Liber abbaci (1202, 1228)

Proposed and solved hundreds of practical problems dealing with trade and commerce, interest, money changing, and so on, but also many non-practical problems. Frequently, he gave multiple methods of solution, so his goal was to teach methods of problem solving and give his readers many different tools.

It is proposed that 7 rolls of pepper are worth 4 bezants, and 9 pounds of saffron are worth 11 bezants, and it is sought how much saffron will be had for 23 rolls of pepper.



Leonardo of Pisa

- There are two men with money. Suppose the first man, having 7 denari from the second, has five times as many as the second, and 1 denaro more. And the second, having 5 from the denari of the first, has seven times as many as the first, and 1 denaro more.
- Leonardo presents four different non-algebra methods and also solves it using algebra.



Leonardo of Pisa

- I divided 60 by a number of men, and each had an amount, and I added two more men, and I divided the 60 by all of them, and there resulted for each $2 \frac{1}{2}$ denari less than that which resulted first. How many men were there?
- Solved by expressing various terms by line segments, drawing squares and rectangles, and finally reaching the equation “census plus 2 roots equal to 48 denari”, which is solved by the standard algorithm.

Jordanus de Nemore (13th c.)

De numeris datis

I-1: If a given number is divided into two parts whose difference is given, then each of the parts is determined.

I-3: If a given number is divided into two parts, and the product of one by the other is given, then of necessity each of the two parts is determined.

II-18: If a given number is divided into however many parts, whose continued proportions are given, then each of the parts is determined.

Medieval Mathematicians

combining theory and practice

- Abraham bar Hiyya (1070-1136), Spain, *Treatise on Mensuration*
- Anonymous manuscript, late 12th century, France, *Artis cuiuslibet consummatio (The Perfection of Any Art)*
- Levi ben Gerson (1288-1344), Orange, France, *Maasei Choshev (The Art of the Calculator)*
- All of these justified the combination of theory and practice.



Abraham bar Hiyya (1070 – 1136)

- Geometry is also as useful for as many matters as arithmetic in worldly matters and commandments from the *Torah*, but is difficult to understand, and is puzzling to most people, so one has to study and interpret it for land measurement and division between heirs and partners, so much so that no one can measure and divide land rightfully and truthfully unless they depend on this wisdom.
- I've seen that most contemporary scholars in France are not practiced in measuring land and do not divide it cleverly. They severely belittle these matters, and divide land between heirs and partners by estimate and exaggeration, and are thus guilty of sin [...] Their calculation might mete out a quarter to the owner of a third, and a third to the owner of a quarter, and there is no greater theft and falsification.

Artis cuiuslibet consummatio (late 12th c.)

The perfection of any art, seen as a whole, depends on two aspects: theory and practice. Anyone deprived of either of these aspects is labeled semiskilled. Truly the modern Latins [by] neglecting the practice fail to reap where they sowed the richest fruits as if picking a spring flower without waiting for its fruit. What is sweeter when once the qualities of numbers have been known through arithmetic than to recognize their infinite dispositions by subtle calculation, the root, origin, and source necessarily available for every science? ... What is more magnificent when once the sides and angles of surfaces and solids have been proved through geometry than to know and investigate exactly their quantities?... We prepare for you therefore a pleasant treatise and delightful memoir on the practice of geometry so that we may offer to those who are thirsty what we have drunk from the most sweet source of our master.



Levi ben Gerson (1288 – 1344)

Because the true perfection of a practical occupation consists not only in knowing the actual performance of the occupation but also in its explanation, why the work is done in a particular way, and because the art of calculating is a practical occupation, it is clear that it is pertinent to concern oneself with its theory. There is also a second reason to inquire about the theory in this field. Namely, it is clear that this field contains many types of operations, and each type itself concerns so many different types of material that one could believe that they cannot all belong to the same subject. Therefore, it is only with the greatest difficulty that one can achieve understanding of the art of calculating, if one does not know the theory. With the knowledge of the theory, however, complete mastery is easy. One who knows it will understand how to apply it in the various cases which depend on the same foundation. If one is ignorant of the theory, one must learn each kind of calculation separately, even if two are really one and the same.



Problem from *Maasei Choshev*

A merchant sells four drugs. The cost of the first drug is 2 *dinars* per *litra*; the cost of the second is 3 *dinars* per *litra*; the cost of the third is 12 *dinars* per *litra*; the cost of the fourth is 20 *dinars* per *litra*. How many *litrans* should one buy of each of the drugs so that the cost for each is the same?



Abacus texts

- In Italy, beginning in the 13th century, the abacus masters taught mathematics to the merchants and their children. They generally taught by means of useful problems – profit and loss, interest, exchange rates. But they included problems that were just for fun.



Examples from abacus texts

The gold florin is worth 5 *lire*, 12 *soldi*, 6 *denarii* in Lucca.

How much (in terms of gold florins) are 13 *soldi*, 9 *denarii* worth?

The *lira* earns 3 *denarii* a month in interest. How much will 60 *lire* earn in 8 months?

A field is 150 feet long. A dog stands at one corner and a hare at the other. The dog leaps 9 feet in each leap while the hare leaps 7. In how many feet and leaps will the dog catch the hare?



Renaissance Mathematics

- 1525 – Christoff Rudolff’s *Coss*. One of first algebra texts in German. Contained numerous quasi-real problems requiring quadratic equations as well as actual practical problems requiring linear equations.
- 1545 – Gerolamo Cardano’s *Ars Magna*. Presented the solution of cubic equations, which required some theory. Problems were mainly quasi-real.
- 1585 – Simon Stevin’s *De Thiende* – presented the theory behind decimals, but also advocated for their use in various trades.



Renaissance Mathematics

- Mathematics used to solve significant problems in physics and astronomy.
- Galileo answered questions on terrestrial motion by experiment and then developed the theory behind his answers by using Greek geometrical methods.
- Kepler used the basic theory of conic sections in developing his ideas for the elliptical orbits of the planets. But he also solved problems in integration by using infinitesimals. As he wrote, “we could obtain absolute and in all respects perfect demonstrations from these books of Archimedes themselves, were we not repelled by the thorny reading thereof.”



The Teaching of Calculus

- Newton and Leibniz used concepts of infinitesimals and indivisibles to solve problems of tangents, velocities, areas, volumes
- The first calculus textbook, by L'Hopital in 1696, determined tangents by drawing infinitesimal triangles and using similarity. There were some basic rules for manipulating infinitesimals, but this theory did not meet Greek criteria.
- Maria Agnesi's text of 1848 also concentrated on solving problems, without much attention to theory.
- The theory of limits, the underpinning of the ideas of calculus, only appears in the early 19th century in Cauchy



The Teaching of Calculus (2)

- Today's teaching of calculus reflects the tension between theory and practice
- Most calculus texts begin with a theoretical study of limits, which leads to a theoretical treatment of derivatives and integrals
- Students learn how to solve problems using the basic algorithms of calculus, generally neglecting the theory
- Yet teachers agree that it is difficult to begin a calculus class with limits. It is better to begin with the problems calculus was designed to solve and leave the theory of limits to later in the course, just as happened historically



Conclusions

- Throughout history, mathematics has been learned by solving problems. Problems are often practical, but in every era, authors proposed “fun” problems as well as “quasi-real” problems, along with actual practical ones. But there has always been theory behind the problem solving techniques.
- Theory and practice depend on each other. One cannot fully understand how to solve problems without understanding the theory. Yet theory alone only appeals to a very limited cadre of students.
- Teachers need to be adept in both problem solving techniques and in theory and must determine the appropriate balance in any given teaching situation