00:00-09:00

The man who came to install our new septic tank had plenty of advice for the Prime Minister – who, sadly, wasn't there, so he gave it to me to pass on, just in case Mr. Cameron ever comes to visit. “It's simple,” said the septic-tank man: “send the immigrants home, send the bankers to prison, increase jobs, cut taxes, increase defense spending, leave Europe. Problem solved.”

I *was* going to say, “it's not that simple” – but it's never a good idea to annoy the person in charge of your plumbing, so I steered the conversation around to the job at hand: “say, why did you mark the new site way down the slope? Shouldn't you dig it nearer the house?”

“It's no that simple,” he retorted, doing that very Scottish thing with this face: “I’d bet yer soil’s clay up by the hoose. Put it there, your tank‘s likely to back up in a couple of years. Doon the way looks more like gravel. I could be wrong; I won't know for certain till I get the digger in and go down a meter or two. Then we’ll see.”

The septic-tank man was doing what we all do, revealing a basic human quality that appears again and again, in lab experiments and real life alike: the less we know about a subject, the more sure we are about our opinions. In the lab of Massimo Piatelli-Palmerini at MIT, people who were *actually betting against the researchers in real money* insisted, at 100-to-1 odds, that more Americans die from homicide than suicide – and that the potato originated in Ireland. It seems that the fewer close dealings we have with a topic, the more important it is to have a quick answer, even if that answer is wrong.

This isn’t necessarily a sign of stupidity; it’s just another way our thrifty brains save on scarce mental resources. After all, when you’re certain of something, you can stop thinking about it – so, yeah, the capital of Colombia is Caracas. Of *course* I'm sure. Next!

When we actually *know* what we’re talking about, though, we willingly admit complexity and uncertainty. It’s a mark of expertise, not indecisiveness, to qualify your statements and avoid simple mantras for success. Roger Federer knows there's more to the game than just keeping your knees bent. In fact, the more you know something, the more fascinating, even baffling, its complexities can be. The great Japanese painter Hokusai said he that only felt he'd finally grasped the rudiments of his art when he reached the age of eighty. Leonardo da Vinci, someone you'd think had ample reason to look back on a lifetime of achievement with pride, instead complained on his deathbed: "was ever anything actually *done*?"

What is it, then that makes someone an expert, rather than just another person with an opinion? "Talent" is the facile answer, but talent really describes a symptom rather than a cause. To be talented is to be so engaged with an activity that what for most of us feels like clumsy wrestling looks like effortless dancing. It is immersion in a subject to the point where you can breathe its new air like a native; hard work that feels like play; observation as intense and nuanced as a lover's. But there's more. There's a particular mode of thinking: probabilistic thinking.

At Dietrich Dörner's lab at the University of Bamberg, experimental subjects get the chance to be what all children aspire to be: kings and queens. In this case, they are appointed benevolent despots of a fictional West African country called Tanaland. This computer-simulated world has many of the interlinked complexities of real life – human population, agriculture, trade, education, industrial development, ecology – but the rulers enjoy a freer hand and more resources than most real dictators ever get.

The result? Almost all of them failed abysmally, producing in only a few simulated years the sort of catastrophic meltdown that would make present-day Zimbabwe seem a paradise by comparison. Even in less demanding theoretical environments than Tanaland, many of Dörner's participants never, *ever* got the idea of how to manage uncertainty; the graphs of their attempts to impose stability, even on the temperature of a single refrigerator, look like the EKG of a heart attack, all spasm and counter-spasm.

Some, though, succeeded – maintaining stability and even improving things slightly. What was their secret? They acted like the septic-tank man – that is, when he's dealing with septic tanks. They asked forensic questions. They made conditional plans, checking their assumptions step-by-step as they went. They described their goals in detail, not as a universal vision. They elaborated ways and means, rather than aiming solely for ends. Most revealing was their choice of language: they used probabilistic terms like *sometimes*, *in general*, *if*, *often*, *a bit*, or *on the other hand*. The failed leaders preferred *always*, *never*, *certainly*, *only*, and *must*. Surprising, isn't it? All those hang-tough terms from the leadership manual turn out to be a prescription for failure. This may explain the careers of several chief executives.

Probabilistic, conditional thinking. Stepwise application of inputs. Simultaneous approaches on many levels. Recognition and analysis of complexity. These seem to be the secrets of successful rulers and indeed of experts in any field. So *if* the Prime Minister ever does come to our house, I'll keep the septic-tank man's advice to myself – but I *will* suggest his actions as the model to follow. After all, the tank he built works perfectly.

09:00-15:00
Charles gave this talk the provocative title, "Mathematicians' Reluctance to Embrace Uncertainty." Well, that's not true, of course: mathematicians *love* uncertainty, because they know that it is at the ambiguous, rippling edges of a problem that all the best discoveries wait to be made. Mathematicians are happiest when grappling with a challenging, fascinating, sometimes frustrating question where they can stretch their mental muscles. As the Danish poet and mathematician Piet Hein said, "Problems worthy of attack prove their worth… by fighting back." I once met a Cambridge post-grad mathematician who was working on the possible symmetries of a group that he cheerfully admitted might soon be proven not actually to exist, thus nullifying four years' work. That seems to me a very close embrace of uncertainty.

So it's not mathematicians who shun uncertainty: no, it's the *rest* of us – students, and teachers, employers and administrators – who shrink from ambiguity and fear the investment of time and money potentially lost in attacking problems that fight back. We are humans: a relatively recent species.

We have not had long to adapt to this world we've built – and our modes of thought often appear better suited to the close horizons and present dangers of the jungle than to the problems of a world-dominating culture.

As I found when reviewing neuropsychology studies for the book, *Bozo Sapiens*, even our basic senses tend to snap to certainty, offering us visual and auditory illusions that can seem as vivid as reality. We look at a pattern of shadows and see a tiger – because, for a vulnerable primate, it is better to be wrong than to be lunch.

We rewrite our memories to be more vivid and more purposeful than real life ever is – not an accurate record of the past, but a past we can live with. We classify the world into stereotypes and assume far greater familiarity with them than our experience actually justifies. We skew what we perceive to fit with what we have already decided we want. And we all believe that we are better than average drivers.

Worst of all, we pass on these rigid habits of thought in the way we teach. Too often, the way the curriculum is designed has nothing to do with helping the student become a true expert in an activity, engaged with all its fascinating ambiguities – but instead aims to make him or her a receptacle for pre-formed certainties. Calculus follows algebra, which follows geometry as the night does the day; vertex A is always in the lower left-hand corner. Oh, and about your question: we don't cover that topic until the second semester. Ask me then.

If I can digress for a moment, I should tell you that I am old enough to have been taught elementary math in three different modes. I started off in the standard arithmetic curriculum that was unchanged since the nineteenth century, where the apparent meaninglessness of what we learned (bring down the next digit of the dividend, plus-five-carry-the-one, nine-twelves-make-one-hundred-and-eight) was relieved by using different colored pencils or having a new tune for each times table. I still know the tunes; I can still multiply. Subtraction, by the way, is red; division is green.

Then educational fashions changed and we were taught Space Age New Math: elementary number theory, truth tables, Venn diagrams, and calculating in base three – an ability that has joined changing a typewriter ribbon, loading a slide tray, and double-declutching a car in my repertoire of obsolete skills.

Then my family moved to England for a winter and I went to village school, where *eighteenth*-century math was still in fashion: field measurements, compound interest, and calculating in pounds, shillings and pence. As an example, if you pay 45 pounds, seven and six for 25 bushels of barley, each will cost you one pound, sixteen shillings, and thruppence… ha'penny. Was this any less useful than 45 divided by 25 in base three? No, in fact it was possibly the best schooling of the three, because by populating its problems with real situations, no matter how far from the reality of a six-year-old, it gave a sense that math is about confronting new things, where *some* of the rules you already know transfer and some don't, requiring new terms, new insights. Water tanks are not barrels, though both hold volumes of liquids… because squares are not circles. Behold! A problem worthy of attack!

15:00-26:00
Mrs. Midwinter, the village teacher, was not, of course, interested in teaching me what real mathematics is like – she wanted to teach me useful skills for when I grew up to be a farmer. And it is this pragmatic reasoning that, sadly, lies behind some of the worst aspects of modern math teaching and the attitudes to math in wider society.

Mathematics should be *useful*, we are told; useful to whom? Whom are we serving? Employers? Governments? School committees? The publishing and software industries? Or are we actually trying to deliver the elusive promise of democracy: a population that can understand and manage uncertainty in a vast, crowded, connected, and increasingly numerical world?

If that's the goal, we haven't gotten there. Public discourse on, say, climate change, shows a frightening willingness to take the uncertainties inherent in the scientific method as a reason to reject the science – as if having an error rate was a mortal sin, or a confidence interval an admission of falsehood.

I was speaking this week with one of Europe's leading breast cancer researchers, and she told me that it is simply *impossible* to convey to politicians – who are, after all, in charge of research budgets – the complexity of cancer as a disease and the uncertainty about treatment outcomes. They want a sure result at a stated cost and the kind of reasoning they are used to – based on lobbying, debate, polls, PR strategy, and communications – leaves no place for assessment of the value of clinical trials by the proper mathematical tools. The data, as far as they are concerned, is just another opinion.

Even professionals can be caught in this confusion. Stefan Senn of the Royal Statistical Society provides a frightening example: health purchasers, physicians, and pharmacists in Canada, the US, and the UK were asked to evaluate four cancer-screening programs to use in their hospitals, assigning relative values for their effectiveness. Here is the evidence they were given to make the decision:

- Program A reduced the death rate by 34 percent
- Program B produced an absolute reduction in deaths of 0.06 percent
- Program C increased the patients' survival rate from 99.82 percent to 99.88 percent
- Program D meant that 1,592 patients needed to be screened to prevent one death.

That Program A looks pretty good, doesn't it? The doctors and administrators certainly thought so: they gave it a score of 7.9 out of 10, well above its rivals. What they didn't notice is that *all four statements describe the same program.* The varying numbers are just different expressions of a single fact; but the relative reduction in deaths – 34 percent! – looked so impressive that they forgot to ask "relative to what?"

Our desire for certainty makes us seize on vivid and defensible – although wrong – positions in preference to the accurate but ambiguous. As a species, we worship the idol of the positive. The Harvard statistician William Cochrane complained that researchers almost always approached him saying, "I want to do an experiment *to show that…*" thus throwing the scientific method out of the window at the outset.

Statistical significance, the basis for almost all empirical research, has been similarly debauched. The *p*-number by which significance is represented, is, as you know, simply the probability that the results gained in the trial could have come about by pure chance. A number below 5% (an arbitrary figure chosen for ease in pre-computer calculation) only *suggests* that there is some mechanism at work rather than random noise. The originator of this measure, Ronald Fisher, was careful not to take it too seriously: after all, a *p*-number of 5% still implies that one in twenty apparently positive results are actually the work of chance. In a study reported in the *British Medical Journal*, merely rolling a die produced results that, if they had represented the outcomes of a clinical trial for cancer treatment, would have been hailed as a miracle cure.

Yet our desire to see pattern and order – our lust for certainty – makes us bend the rules. Scientific journals publish more positive results than negative ones, despite the fact that either result is equally valid and useful. There is always a temptation, therefore, to tweak or selectively sample a data set to pass that magic 5% threshold. Even without fraud, there is danger: we have a serious outbreak of measles going on right now in the UK that can be traced to a misinterpretation of statistical significance producing a false correlation between the measles-mumps-rubella vaccine and autistic spectrum disorders. Journalists (and, unfortunately, some scientists) are far too ready to talk about "proven with 95% probability," confusing the 5% chance of a result given an assumption of randomness, with a 5% chance of randomness given the result. It's like saying that the 95% chance that I will carry an umbrella if it's raining is the same as a 95% chance that it will rain if I carry my umbrella.

"Standards are falling day by day," said Seneca the Elder – two thousand years ago. Plutarch complained that students are noisy and spend the whole lesson chatting to their friends. What I'm saying may sound like grumbling about how young people today don't know any grammar.

But – really – this is not trivial. An innumerate populace is a public danger. If we are incapable of assessing the results of science, it becomes indistinguishable from religion. If we cannot gauge uncertainty, risk, confidence, likelihood, moral hazard, and the other probabilistic factors that govern our indeterminate world, we will have thrown away the most valuable tools that mathematics offers to practical life.

People have complex decisions to make: about their future careers, about their money, about their government, about their environment. At the same time, they are being misinformed and misguided by *other* people, who either have vested interests or are too lazy to check the numbers. Darrell Huff's book *How to Lie with Statistics* was first published in 1954. It's short, simple, amusing, convincing – yet nearly sixty years later, you can find an example of every one of his catalogue of intentional and unintentional misinformation in this week's news.

We live in a society with not just the freedom, but the seeming *compulsion* to make known our opinions – whether we know what we're talking about or not. In the US in 2010, there were 16,000 bachelor's degrees awarded in mathematics-related topics – and 81,000, *more than five times as many*, in communications and public relations. If we want the signal to be heard above the noise, we have our work cut out for us.

26:00-33:00
How do we deal with this? We are given twelve years to teach children mathematics, and the usual result is that they can't use it and they don't like it. We have to do something – and, as Einstein pointed out, "you cannot solve current problems with current thinking – current problems are the *result* of current thinking."

Well, we can start by lying less to our children. In March 2012, the Royal Institution announced the results of a survey of the career plans of 1,000

children aged 6-16; the results showed "a worrying disparity between children's career aspirations and their study choices."
The top ten "dream careers" were revealed as:

- 1. Professional Athlete
- 2. Performer
- 3. Secret Agent
- 4. Fire fighter
- 5. Astronaut
- 6. Veterinarian
- 7. Doctor
- 8. Teacher
- 9. Pilot
- 10. and Zoo Keeper.

The slightly self-congratulatory press release that accompanied the results said how "incredibly encouraging" it was that five of the top ten were "exciting careers in STEM" – vet, doctor, pilot, astronaut, and zoo keeper (although one wonders whether the average school child's image of a pilot is of using a flight calculator or doing aerobatics, or of a zookeeper is performing DNA electrophoresis – or cuddling baby pandas).

Nevertheless, this was good news; the only problem was that 49.4% of the children thought that STEM subjects were too difficult and boring to study and a further 15% though that there was no point in them because one could only succeed with straight A marks. The Royal Institution decided that its challenge was to convince children that STEM subjects were actually fun and would help them achieve their "dream careers."

Is this likely? The US Bureau of Labor Statistics publishes some very helpful tables with, since it represents a larger market, more stable data than we might get from the UK. Adding up the top five of the Royal Institution's dream careers, professional athlete to astronaut, the BLS estimates an average total of 5,485 new jobs in these careers opening each year for the next decade. The annual number of people entering the US job market is 1.08 million. So, all other things such as having the requisite sporting and performing talent being set as equal, the chance of getting a job in *any* one of those five – being willing, say, to settle for firefighter despite having your heart set on being a secret agent – is one in 200. And even if you did get them, many of these jobs are, sadly, ill-paid and unfulfilling – and, recent research at the University of Queensland suggests, the top three of them usually mean a shorter than average lifespan.

As it happens, the top ten US job categories, with 50,000 or more new jobs in each, a higher than 29% growth rate, and an annual salary over the national mean, are *all* dependent on STEM skills. Every single one: software developers, financial advisers, dental hygienists, cost estimators – even air-conditioning engineers and concrete finishers (who, as my septic-tank man can tell you, have to make complex volume estimates incorporating thermal expansion and relative humidity). We don't have to pretend that every child can become an astronaut; the facts are there. Wouldn't it be best if we provided children with the *generic abilities* to assess and work with such data, and helped them to use it to make their own decisions?

Life has to be lived from the beginning, but it only makes sense when seen from the end. "Dream careers" so often become illusory careers. We make a sad mistake when we promise that studying math or other sciences is bound to lead to a fulfilling destiny; that's false certainty. We do better when we present the odds, the likelihoods of life – and draw, together with the student, the likely conclusions about what skills are necessary. Otherwise we end up with a population of unfulfilled, unskilled, and unhappy young people.

Besides, there is an essential flaw in the argument, "learn this material and you will have a prestigious, high-paying, glamorous career." It's delivered by that sneering voice from the back row of the classroom: "Oh, yeah? *You* learned this material – and you're *a math teacher*!"

Instead, the message could be, "learn this and gain mastery over the unexpected, avoid being made a victim, predict the future, sort out what you really know from what you only believe, find true love, and use your mind as the subtle knife it was designed to be!" That sounds a little more exciting.

33:00-41:00+

As you have probably guessed, I am not a mathematician – although I grew up surrounded by them and have some sense of the specific qualities of mathematical thinking. My training was as a historian – but the two subjects are not all that different from each other. Both seek methods by which to isolate coherent instances from chaotic ones, and to devise and test statements that have relevance in more than one instance. The only difference is that history unfortunately includes no proofs, which is why historians are so excited by counter-examples.

It was history – which is, after all, no more than the would-be science of human events – that led me to an interest in one neglected aspect of mathematics that I think could have the greatest relevance for the greatest number of students: probability, which one could call the formal study of fate.

Mathematicians tend to glaze over at the mere mention of probability: it's too mechanical, or it's too controversial, or it's too compromised by grubby real-world data ever to match the dazzling formal beauty of the purer disciplines. That may be so, but these same qualities can make it fascinating and accessible to students. People can get to grips with this mechanism: every poker-player understands contingent probability; every baseball player can re-calculate his or her batting average on the way down to first base. Bayes' formula, the basis of inductive probability, is far easier to remember and use than the quadratic formula, yet its repeated, mechanical use reveals wonders, from how likely a criminal defendant is to be guilty, to how likely it is that the sun will rise tomorrow.

Probability is controversial, in that it contains unresolved disputes, but many of these are fruitful – they reveal flaws in perception and definition that, once discovered, make for better and more careful thinking. Take this demonstration, which the mathematician James Gleason used to perform in his class:

[perform it]

You see? Even in the first flip, we are making an assumption about a physical system – that it's fair, or fair enough (although the mathematician and magician Persi Diaconis can flip a fair coin so that it *always* comes up heads). In the second, we have a question of information: what exactly are we betting on? How many cases? All those semantic side-issues that so often come up in the math classroom have real relevance here.

In probability, we can quickly get students exposed to that fruitful area where mathematics doesn't automatically provide the right answer – indeed where mathematics may not yet have a right answer. A class that has worked through some of these simple questions will be far less susceptible to being blinded by spurious statistics.

Probability questions also rapidly accelerate students from simple problems to baffling, exhilarating answers. The most famous of these basic mind-stretchers is what's called the Monty Hall, or game show problem. How many people here know it?

[If they don't, demonstrate it.]

If you're saying to yourself; "I see it, but I don't believe it," you are not alone.

If you want to get any real-life situation into the form about which you can calculate, you need to make likelihood estimates. Enumerating cases, isolating variables – classical probability. If you want to face reality without superstition or false belief, you need to draw empirical inferences from sequences of events – Bayesian probability. We have a passionate need to find out how the world behaves, and how we come to find out about it – how trustworthy our little maxims and heuristics are. Children do this already and they already know it's important. They just don't know that it's math.

And is it relevant? Oh, yes. There are people serving life terms in jail because a jury (and most judges) can't tell the difference between a one-in-a-million DNA match and a million-to-one chance of guilt. People have died because doctors don't accurately compare disease rates and false-positive test rates. Probability, applied to big data, has helped bring some of the world's worst human rights violators to justice. Medicine, law, finance, weather, farming, war – if it's human, it's a topic in probability. After all, most of life is *somewhat*, *to a degree*, *only if*, *when not*, *perhaps*; we need the numbers of uncertainty to make sense of a chancy, risky world.

41:00+-48:00
The type of measurement determines the nature of the thing measured. If we measure hospital performance by the number of avoidable deaths, hospitals will class all incoming patients as moribund. If we measure mathematics teaching by rankings on exams, we reward competitive exam taking – which isn't mathematics. What we tend to put on those exams (for the convenience of the examiners) isn't mathematics either – nor is it particularly useful in life. As we move further toward online teaching and computer-marked work, this tendency to concentrate on what we can measure is going to get more pronounced.

But you learn French by living in France. You learn soccer by spending every summer evening kicking a ball with your friends. Mozart may have seemed a prodigy at fourteen, but he had spent the previous ten years in constant contact with the best musicians in Europe. Why, then, should you learn math from a computer, or from someone reading out of a textbook? We need to embrace uncertainty, not just in life and in discovery, but in the actual process of teaching.

In a small room in the Science Center at Harvard, a mathematical researcher leans back, exhausted but exhilarated. She and her colleagues have just made an astonishing discovery: there are numbers between zero and one – a *kazillion* of them – in fact, there are a kazillion or more numbers between any two numbers you choose. Impressed? Well, these pioneers *are* only five years old.

They are part of the Math Circle, an informal, collegial program of math learning, open to every child or adult willing to give up three hours on

Sunday or an hour on weekday nights to come in and attack some worthy problems. The kindly teachers at the front – asking those questions that make learning seem like remembering – are my parents, Robert and Ellen Kaplan, who started the Math Circle twenty years ago… or one of the twenty young mathematicians, usually Harvard or MIT graduate students, who have joined in the hopes of making learning math more like doing math.

This is not some hot-house exam-prep crammer, where students race against the clock and each other to shout out the right answer; nor is it pat-a-cake math, with watered-down, artificially sweetened problems. It's the real thing, pursued in a lively but respectful environment where problem-solving proceeds collectively, prompted in Socratic fashion by the teachers. It is not a whizz-kid seminar: there are no admissions requirements. It is not an accelerated path through the usual curriculum. It is often the case that a class spends three or four successive sessions on the same problem.

But the problems are packed with significance; they are the pivotal theorems where number theory, geometry or calculus takes a sudden skip and turns down a new avenue – every proof is also a revelation. And the class creates that revelation for itself. Because the teachers keep opening the right doors in front of the conversation, the students themselves generate all of the essential concepts, inventing their own terminology and sharpening their proofs in debate.

"How do you know that? Is this a valid assumption? Is this term equivalent to that in all cases? What if we turn the technique on its head?" This is exactly what real mathematicians do, and is good training for any discipline. But its greatest reward, unique to math, is this: when you have gone through all that, you come out with something *certain* – newly revealed, astounding, unbelievable, yet permanently true. To achieve this magical result, though, the teacher has to be willing to tolerate a lot of uncertainty in the process.

And what are the children getting out of it, asks the government inspector and the future employer? Well, they are getting things they won't forget in the hour after their last exam in the topic. Not just familiarity with some of the most challenging, fruitful problems, but a knowledge of how problems in general need to be observed, understood, and attacked.

And more than that: how to make and defend a point persuasively, and how to accept a counterexample with good grace. How a room of different personalities and abilities can work together, drawing the best from each to the benefit of all. These are skills that will serve them well, whether they become astronauts, actors, or air-conditioning technicians.

48:00-50:00
Every teacher has heard it: "Why are you telling us all this? Just give us the answer" – that despairing suicide note from the future. "Is this going to be on the test?" The real test, of course, is life: and the answer is a prepared mind.

The Russian army considers mere "yes" or "no" to be too vague a statement and insists that soldiers answer either *tak tochno* (exactly so) or *nikak niet* (not nohow) – a degree of certainty useful only for military obedience. In the real, probabilistic world, there are no such absolutes to hide behind. We reason, and examine our reasoning, not because we expect to achieve certainty, but because some forms of uncertainty are better than others – some explanations have more meaning, more connections, less entropy.

What non-mathematicians, like me, need from mathematics, therefore, is something more subtle than "exactly so" or "not nohow." We need a handle on the calculable aspects of human existence. We need to know when we are being taken for fools, or offered some unfair bargain. We need to know when news is only opinion, or when the promises of our governors are mere fantasy. We need to navigate the broad seas between the extremes, checking our course against the constant guides that mathematics offers – and, of course, we hope for that same thrill of abstract beauty and sudden revelation that mathematicians live for.

As Francis Bacon put it, four hundred years ago: "If we begin with certainties, we shall end in doubts – but if we begin with doubts, and are patient in them, we shall end in certainties."