Report to the
Bill & Melinda Gates Foundation

Difficulties in and about Mathematics Education and their Solutions

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Prepared for the Center for Curriculum Redesign by:

Charles Fadel

with review by

Brendan Kelly, Harvard University,
Neil Marshall, New Zealand Qualification Authority,
and Emma Smith Zbarsky, MathWorks

Note: this is a consulting report with educated opinions,
not a research/academic paper.

“Those in the game see less clearly than those outside” (Liu Xiu, ~900 AD)
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Preamble: Who is CCR? What are their credentials in Mathematics?

The Center for Curriculum Redesign (CCR) is a Boston-based non-profit whose mission is to “Make education more relevant.” In the past decade, it has funded two conferences\(^1\) and two colloquia on Mathematics for discussion and synthesis of recommendations to OECD.\(^2\) These efforts inspired the 2017 paper “PISA 2021 Mathematics – A Broadened Perspective,” which then led to the 2018 “PISA Mathematics Framework.” They were eventually partially implemented in PISA 2021.

Andreas Schleicher, Director for Education & Skills at the OECD, has commended CCR’s efforts. “I would like to commend the Center for Curriculum Redesign (CCR) for the work it has achieved over the last four years, and its contribution to the advancement of Mathematics PISA 2021... CCR has doggedly pursued an agenda of relevance and modernization of education standards and assessments, applied to Mathematics in this case. As you know, PISA’s future and continued success will closely depend on our ability to rethink “What should students learn for the 21st century?” – CCR’s seminal question.”

Since then, it has, in partnership with Australia’s curriculum authority (ACARA), created a modernized set of standards for preK-9.

Michele Bruniges, Australian Department of Education and head of the PISA Governing Board, commented:

“CCR’s work is critical in the unique value it adds to curriculum redesign by being:

- Non-political – not beholden to local constraints.
- Non-dogmatic - working with experts from a broader range than typical.”

Janet Davis, then-Curriculum Director at ACARA, added: “You are 5-10 years ahead of everyone else, where we must get to quickly...”

It is now creating, in partnership with the OECD, an assessment option called “Primo” for jurisdictions that wish to advance the mathematics abilities of their populations by

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\(^1\) The first one of which, in Stockholm, was funded in part by the Gates Foundation: [https://www.gatesfoundation.org/about/committed-grants/2012/08/opp1068155](https://www.gatesfoundation.org/about/committed-grants/2012/08/opp1068155)

exploring modern mathematics subjects (Algorithms, Bayesian probabilities, Complex systems, Game theory).

The Gates Foundation also recently provided CCR a grant to do an analysis of Learning Outcomes for several disciplines including Statistics/Probabilities.  

**Difficulties**

For many OECD countries including the U.S., Mathematics education should fulfill two main goals:

1. General mathematical literacy of its population at large (problem-solving and interpreting).
2. Labor needs in STEM occupations and in advanced, innovation-rich fields such as Data Science and Computer Science. The appendix contains two tables “Uses of Mathematics by Various Professions,” and one with “Examples of Occupations that can Benefit from Modern Mathematics.”

But these two main goals already pose contradictory challenges for curricula (e.g., math for vocational students or math Olympiads? Or said differently: education for everyone, or for the elite? Clearly it should be for both, so **if for both, how?**).

Furthermore, the two main goals are clouded by a significant number of other desires and wishes. The “goals and functions” table in the appendix showcases the complexity.

Lastly, the present system’s **inertia** prevents agility of adaptation in spite of NSF/CBMS recommendations dating back to...1982! such as the introduction of more modern branches of Mathematics at a time of pent-up demand and far greater need for both general math literacy and STEM occupations:

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3 https://www.gatesfoundation.org/about/committed-grants/2021/10/inv031028
4 https://curriculumredesign.org/intro-stats-probs/
5 “Gullible citizens are a demagogue’s dream...almost every political issue has a quantitative aspect” (John Allen Paulos, Temple University, author of “A mathematician reads the newspaper”)
7 NSF Mathematical sciences 1982 “what_is_fundamental_report”: “more emphasis on estimation, mental math...” “less emphasis on paper/pencil execution...”; “content in... algebra, geometry, pre-calculus, and trigonometry need to be... streamlined to make room for important new topics;” “Discrete Mathematics, statistics/probabilities and computer science must be introduced.”
Inherent Complexity of Mathematics

Unlike other disciplines, Mathematics has its share of soured former students as it is considered difficult to the point of derision – up to loathing - by vast swathes of the population. For the sake of sticky messaging via levity, here’s a sample:

8 https://xkcd.com/1050/
Slightly more analytically, a quick sentiment analysis of adults\(^9\) shows the negative opinions in the Twittersphere – and note the progression, as Math becomes more complicated (Arithmetic $\rightarrow$ Geometry $\rightarrow$ Algebra), with, notably, Arithmetic and Statistics being perceived less negatively:

![Sentiment Analysis Diagrams](image)

This report will examine what makes improving Mathematics education difficult by looking at four facets, two intrinsic and two extrinsic, to mathematics as a discipline:

1. **Intrinsic: verticality**
2. **Intrinsic: abstraction**
3. **Extrinsic: faculty (Higher Ed, and K-12) and students**
4. **Extrinsic: assessments**

**1. Verticality:**

Disciplines are categorized as more or less “vertical,” meaning that their knowledge structure is more or less hierarchical vs horizontal. Importantly, the hierarchical ones

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\(^9\) Done by the author in 2013, but we doubt the situation has evolved noticeably.
are data-centric (i.e., mathematics), while the horizontal ones are text-centric (i.e., literature).\textsuperscript{10}

Mathematics is at the end of the scale of verticality, which means it is very \textit{sequential}: Arithmetic, then Geometry & Algebra, then more advanced Algebra (and its various forms, including discrete/computational and statistics/probabilities). One other route would be to think about learning progressions, e.g., Counting \(\rightarrow\) Additive Thinking \(\rightarrow\) Multiplicative Thinking \(\rightarrow\) Proportional Reasoning \(\rightarrow\) Exponential Thinking. This diagram below shows a very small portion of learning progressions:

\textsuperscript{10} Maton, K. (2009), \textit{Cumulative and segmented learning}; and Bernstein, B. (1999) \textit{Vertical and Horizontal discourse}. 
The drawback of verticality is that missing a step endangers the ability to go further, as there are no possible refuges; whereas in literature, one can read Hemingway without having read Dickens first, and vice-versa.

This verticality also implies three levels of potential complexities for the teachers:

- Macro level: the S-curves of abstraction levels displayed below
- Meso level: within a given cluster (say, Proportionality), there are multiple pedagogical pathways to proceed
- Micro level: within a given pathway, there are multiple types of errors made by the students (from arithmetic mistakes to false reasoning)

A cogent curriculum would analyze the complexities at all three levels then devise detailed recommendations for teachers.

2. Abstraction

Mathematics is inherently a stepwise voyage from the concrete into the abstract, requiring to “pause and digest fully” at each step, which does not happen at the same rate for all students.

From a number of physical objects to a digit representing the objects’ count, from number to variable, from variable to function, from single variable to multiple variables, from linear to non-linear coefficients of functions, from multiple variables to linear transformations, from linear to non-linear-transformations, etc. etc. to Eigenvalues and beyond (cohomology, anyone?), each step represents another hurdle for the student.

Furthermore, abstraction, while being a natural capability of humans, is generally much more developed on the side of emotionally evocative stories than it is on the side of data.\(^{11}\)

Evolution has equipped humans to deal with relatively simple numbers, perceived as decreasing increments,\(^{12}\) which means that even number sense needs to be “linearized” for a young child (and number sense remains a complexity into adulthood: can you visualize a trillion-dollars’ worth of bills going to the moon and back?).

\(^{11}\) Kahneman et al, “Thinking Fast & Slow”
\(^{12}\) Dehaene, Izard et al, 2008; and Siegler and Opfer, 2004
Per Stanislas Dehaene in “Number Sense”:

“Mathematics is a language, but that language does not appeal to classical language areas of the brain. Mathematics builds upon ancient, non-linguistic foundations: core knowledge of number, but also of continuous quantities, space, time...shared with many other animal species. Such core knowledge can emerge in the absence of any visual experience, and developmental evidence suggests that the corresponding brain circuits are available from birth.”
Evolution has also prepared us to deal with shapes and uncertainty\textsuperscript{13} as well as quantity (\textit{Let us note in passing that uncertainty, essential for survival, is underappreciated by mathematics’ education.})\textsuperscript{14}

Developmentally appropriate

\begin{center}
\begin{tikzpicture}

\node (calc) at (0,0) {Calculus, advanced Algebra, Topology, advanced Probs/Stats, etc.};
\node (alg) at (0,-2) {Algebra and Logic};
\node (quan) at (-2,-4) {Quantity (Arithmetic)};
\node (shape) at (0,-4) {Shape (Geometry)};
\node (uncertainty) at (2,-4) {Uncertainty (Probability)};

\draw[-stealth, thick, color=gray] (alg) -- (calc);
\draw[-stealth, thick, color=gray] (alg) -- (quan);
\draw[-stealth, thick, color=gray] (alg) -- (shape);
\draw[-stealth, thick, color=gray] (alg) -- (uncertainty);

\end{tikzpicture}
\end{center}

\textit{Innately : manipulating quantity, shape, uncertainty}

Adapted from Stanislas Dehaene, CNRS

All this to say plainly that, \textit{beyond middle school mathematics}, Algebra is the rough analogue of Philosophy on the language side – not an easy endeavor for many, hence the cartoons above.

\textsuperscript{13} “\textit{One may even say, strictly speaking, that almost all our knowledge is only probable; and in the small number of things that we are able to know with certainty, the principle means of arriving at the truth - induction and analogy- are based on probabilities.}” Pierre-Simon Laplace, \textit{Théorie analytique des probabilités}, 1825

\textsuperscript{14} \url{https://curriculumredesign.org/wp-content/uploads/Why-are-even-basic-Probabilities-so-difficult-to-learn-Charles-Fadel-CCR.pdf}
The situation is further aggravated by the fact that, for most professions, middle-school mathematics is sufficient for most occupations, per the diagram below:

To situate the graph’s x-axis; Level 6 is:
- May require considerable translation from verbal form to mathematical expression
- Generally, require considerable setup and involve multiple-step calculations

This means the old adage of “use it or lose it” is prevalent for professions, and even PhDs in mathematics can no longer read their dissertations if they have not practiced the corresponding mathematics for a number for years.\(^\text{16}\) (It is akin to asking a fighter jet pilot to climb back into an F-35 cockpit after flying an ultralight plane for 40 years).

\(^{15}\) Private communication, from Dr. Merrilea Mayo’s analysis of ACT WorkKeys: https://www.act.org/content/act/en/products-and-services/workkeys-for-employers/assessments.html

\(^{16}\) Anecdottally, my uncle: PhD in Probabilities (Hidden Markov Models) in 1980, unable to read his dissertation’s first page now.
However, this occupational need does NOT correspond to the need for general mathematics literacy for daily use, which will be discussed in the Solutions section of this report.

3. Faculty (and students)

a. Higher Ed Professors

Given their expertise and status, Higher Ed professors are solicited by jurisdictions to help design and develop standards-curricula, and as such bear a high degree of responsibility in the results. Like all humans, they are prey to psychological biases, in spite of their vehement assertions of being guided only by logic.

- “Groupthink” - The typical standards design exercise consists in comparing standards from various jurisdictions around the world, which reflect each other with very little innovation. And the OECD’s surveys\textsuperscript{17} neglect to mention modern areas of mathematics, even as a data-gathering exercise, \textit{because the surveyors themselves do not know them},\textsuperscript{18} and over-emphasize their areas of comfort:

\begin{itemize}
  \item \textsuperscript{17} OECD 2030 project, Document number EDU/WKP (2022)6 page 17; OECD Education Working Papers No. 268
  \item W. Schmidt et al, 2022 https://dx.doi.org/10.1787/07doeb7d-en
  \item In the OECD document cited above in footnote, one will note the overemphasis on Stats (the surveyor’s specialty area), mentions of exponentials and geometry (per CCR’s recommendations), and a messy mishmash of Boolean algebra, coding, and “computational thinking.”
\end{itemize}
- Traditional thinking - Most Mathematics experts reflect the following,
  - Theoretical bias - many disciplines are dominated by academia, whose professions shield from real-world pursuits. Furthermore, mathematicians are extremely comfortable with the abstraction described above, as shown in their psychological profile:
This theoretical bias is shown in the relative disdain for applied branches by excluding them for what they consider Mathematics, calling them “Mathematics and Statistics,”20 or “Mathematics and Mathematical Sciences.”

- “Prove it to me” bias - the “Establishment,” which spends approximately $340b/year21 on Mathematics education just in OECD countries, does not believe it needs to justify itself for its decades-long mediocre results, while Innovators have to go through hoops to be heard.
- “Purity” bias - Mathematics has given the vibe of itself the “purest,” which can be traced back to a Judeo-Christian concept of purity,22 depicted humorously in the cartoon below.

19 Private communication from HBDI: https://www.thinkherrmann.com/hbdi
20 Ironically, statisticians feel honored by being mentioned explicitly, while that mention actually implies that they are not “real” mathematicians.
21 Private communication, from Andreas Schleicher, OECD
22 Presentation: “Mathematics and the drift towards Purity” – Sverker Lundin (University of Gothenburg)
Lack of familiarity\textsuperscript{24} with modern subjects and topics of mathematics\textsuperscript{25} - this is linked to the point above about a theoretical mindset, and the deep discomfort at discussing something they are not familiar with.

b. K-12 Teachers
Given their impact on K-12 students, teachers are on the frontlines.

- They are served a set of standards to meet that is generally too large to insure non-rote coverage. As a result, they are unable to focus on developing Competencies (critical thinking, growth mindset, etc.) and transferable Core Concepts such as “deceiving then explosive” for Exponentials, rather than only the inert mathematics formulation:

\begin{center}
\includegraphics[width=\textwidth]{xkcd_purity.png}
\end{center}

\textsuperscript{23} \textit{xkcd: Purity}

\textsuperscript{24} That same lack of familiarity exists with the curriculum designers, and assessment designers (as demonstrated by ETS in its PISA 2021 rework).

\textsuperscript{25} “There is an assumption that a PhD in mathematics qualifies someone to teach/design curriculum for students with modern needs. This is absurd. A successful algebraic topologist might have little knowledge about what math a person studying biology needs to know.” - Brendan Kelly, Director of Introductory Mathematics, Harvard University
This problem of scope has been decried over the decades, but even Covid-19 has not had a sufficient impact on forcing jurisdictions to significantly reduce scope. The root cause is the interplay between assessments for higher education entrance requirements, and Higher Ed experts’ traditionalism (next section).

Teachers are not trained to embed Core Concepts and Competencies in their lesson plans. For that matter, Mathematics as a discipline has done a very poor job at differentiating between content and concept, per the exponential example above. This is a huge ramp-up, needed at the time when there is a deficit of teachers that ranges from worrisome to catastrophic depending on the jurisdiction.

Teachers are all the more untrained in areas of modern mathematics that fall outside their traditional training and enshrined in assessments’ foci.

Teacher organizations/unions are a major contributor to the status quo, for all the reasons described above.

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26 In the U.S., Achieve the Core made some light-touch recommendations, as detailed in the CCR database.

27 Other disciplines like History have indeed done a better job: https://www.oecd.org/education/2030-project/about/documents/Learning%20progression%20in%20history%20-%20Zarmati.pdf

It took CCR and ACARA three years of work to extract the Core Concepts. See left bar here: https://mathstandards.curriculumredesign.org/

28 For instance, in the Netherlands, the task of modernizing the standards was given to the union, under the theory that “teachers know what math is needed by society.” They do not, and the attempt failed.
c. Students
Evolution has wired us to be economical in our use of brain energy, as this organ represents 20% of the energy consumption of the body for 5% of its mass. One could quip that we are “lazy by good evolutionary design.” As a result, we – as students – are constantly assessing “why pay attention?” Appealing to the logical “system 2” requires motivation - relevance - which translates into perceived usefulness thus applicability.29

4. Assessments
Assessments in Mathematics have featured the conflation between algorithmic procedures and logical reasoning, as even Mesopotamian scribes were selected based on their ability to handle then-useless quadratic equations as a sorting mechanism:

29 Which is likely why Stats/Probs and Arithmetic rank higher in sentiment analysis
30 New Yorker cartoon by Robert Weber
a) In the U.S., university entrance requirements have been dominated by the College Board’s SAT tests, and to a smaller extent the ACT. It has been demonstrated\textsuperscript{32} that they are a proxy for \textit{tenacity}, which is a predictor of success for (selective) university entrance counselors. This system is further entrenched by the proclivity by university admissions to sort by the number of AP courses taken by a student.

The irony of this situation is two-fold: Mathematics professors do not ask for this sort of screening (the admissions staff does), and many universities force students to retake the AP courses rather than get equivalency (we will propose why: profit motive, and/or lack of trust).

This has the enduring effect to freeze in place the K-12 standards, and assessments, per the diagram below, with heavy consequences:

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\textsuperscript{31} Source: Victor Katz

To elaborate on the diagram above: the needs for life and work are changing and not always crisp, while university entrance can be more specific and easier to target. So the university entrance dictates the types and content of assessments, which in turn dictate the standards. Common Core,\(^{33}\) which had the benefit of harmonizing the hodgepodge plethora of state standards, is part of the problem by further entrenching the closed-loop system.

So, Algebra++ is the modern-day equivalent of quadratics for the Mesopotamians. This has deleterious effects on DEI, whereas other disciplines are at least as important for life success.

**Solutions**

1. **To problems induced by Verticality:**
   a. Design a cogent curriculum which analyzes the complexities at all three levels (Macro, Meso, Micro) then *devises detailed recommendations and deep professional development for teachers*. This needs to be achieved by a credible, brand-rich team that has *modernity* in mind as a singular goal and is shouldered by a significant buy-in campaign.
   b. Design an adaptive, EdTech-enabled pedagogy that, married with didactic instruction and *EdTech-based peer-to-peer tutoring (both leveraging social learning)*, allows students to master the verticality (and abstraction) at a personalized pace; *note that this approach needs to be carefully researched as to the best blend of possibilities.*

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\(^{33}\) Unfortunately, Common Core had to deal with the Realpolitik of the times and had to accept two false axioms: that teachers were not retrainable and that university entrance was not changeable.
2. To problems induced by Abstraction:
   a. The cogent curriculum in 1.a. needs to be made:

Adaptive Learning’s Promises

- Self-pacing
- Learning time reduction
- Additional gains with A.I.?
i. Applicability-related by adding the two applied branches discussed above.

ii. Real-world-centric, with projects accompanying didactic instruction.

iii. Core Concepts- and Competencies-infused as well, explicitly, for Transfer.34

iv. Offered as two (if not three?) pathways (see Appendix):
   a) STEM college-bound: Produce Mathematics
   b) Non-STEM college bound, and non-college-bound: Interpret/Appreciate Mathematics

This example below can help clarify this approach, as well as the appendix on pathways:

<table>
<thead>
<tr>
<th>Level in High School</th>
<th>Produce</th>
<th>Interpret</th>
<th>Appreciate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 3 (STEM-bound)</td>
<td>Produce</td>
<td>Produce</td>
<td>Interpret</td>
</tr>
<tr>
<td>Level 2 (College-bound)</td>
<td>Produce</td>
<td>Produce</td>
<td>Appreciate</td>
</tr>
<tr>
<td>Level 1 (VET-bound)</td>
<td>Produce</td>
<td>Interpret</td>
<td>Appreciate</td>
</tr>
<tr>
<td>Example</td>
<td>Exponentials</td>
<td>Quadratics</td>
<td>Cardioid, cycloid etc.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Graph Theory</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Infinities &gt;</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>infinities</td>
</tr>
</tbody>
</table>

Note that this solves the issue of relevance to students, but it also means that existing standards will have to be rebalanced.

3. To problems induced by Faculty35

   Higher Ed academics

   a. Choose open-minded, polymathic academics, recruited from leading organizations, who understand the usage of their discipline and pair them with Users (engineers, social scientists, etc.) throughout the development of standards/curriculum. Furthermore, the Users must be constantly part of the process (not pop in and out as is usually the case) and drive the conversation.

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34 ‘This is an incredibly underleveraged idea. Mathematics provides an important tool kit for students to better understand the complexity of the world. However, partner disciplines fail to bring mathematics into the classroom/curriculum in meaningful ways. Just like we have “reading across the curriculum” or “writing across the curriculum” we should have “data/math/modeling across the curriculum.” A social science teacher should be supported to talk about redistricting from a quantitative perspective. The timing of the lesson should be in conjunction with and supported by lessons from the math classroom. The same could be done with science classes. Imagine if at the same time students were talking about chemical reactions in a chemistry class, students were coding reaction simulations in a math class?” - Brendan Kelly, Director of Introductory Mathematics, Harvard University

35 This is the process used by CCR.

36 Themselves carefully chosen, i.e., CBMS and OECD, not just NCTM and MAA
b. Involve a small, dedicated design team. The old joke about academia applies: “In academic decisions, one dissent out of 30 is a tie,” which leads to regression to the mean and abandoning the more daring and innovative top 5% of ideas.

c. Socialize very carefully in concentric circles for consensus-building but starting with the organizations themselves if possible. Avoid tumult at all costs.37

d. Alternatively (or perhaps simultaneously), devise a flanking strategy via a point of high leverage (OECD/PISA for instance).

K-12 teachers

a. Recruit and reward socially and financially.
b. Train, train, train and retrain38 - for expertise in present content; infusion of concepts and competencies; eventually, in new branches.
c. Augment via technology per 1.b. above.

Compounding the difficulties, the mathematics department staff of jurisdictions is composed of the two categories above and suffers from limitations such as:

- Insufficient/non-existent expertise in new mathematics branches
- General unwillingness to accept help from outside their jurisdiction
- Fits and starts process: no action for a decade, followed by a rush during a year or two, then no action again for a decade.
- Pressures from politics (parties, parents, unions)
- Well-meaning but useless attempts to ask students what they want (no, they don’t know any better!)

All of which force a deep inertia. And ironically, the success of Common Core (even if not admitted by many states) could become another such inertial force if allowed to calcify. The appendix contains an analysis and a crosswalk of CCSS and CCR’s standards.

Luckily, Mathematics does not change constantly: there is a pent-up backlog of changes that should have been made fifty years ago that are overhanging. This situation can be resolved within a decade and merely tweaked every five years or so.

Additionally, in some jurisdictions, rather than make explicit room for Financial Literacy and/or Computer Science, the Math teacher is burdened with these disciplines. Using compound interest as an example of exponentials is one thing, teaching full-

37 Which means the US may not be the easiest place to start given its highly fractious nature.
38 The training must be made mandatory; as one of the reviewers, Neil Marshall from NZQA, stated: “good PLD is available but the only ones to engage are the ones who do not need it.”
fledge financial literacy is another. Similarly in computer science, learning Algorithms is Mathematics, learning coding is not.

Last but not least, there is little communication to students explaining relevance at every step of the way.

4. To problems induced by Assessments:
   a. Run a series of workshops with the National Association for College Admissions Counseling (NACAC) to understand their needs and devise alternative sorting mechanisms that can scale; for instance, valuing other branches of mathematics, other disciplines, or other mechanisms, as demanding and worthy of being a proxy for tenacity. (Philosophy for one!)

**Political Difficulties in the U.S. Context**

The U.S. suffers from the following additional difficulties:

- **The reflex by conservatives to oppose anything new or different (as witnessed in the UK after Blair).** Appropriate collaboration is essential to avoid miscommunication.
- **The fundamental inequity of local school funding:** to solve for DEI, there is a need for significant funding for poor schools to afford:
  - ample additional training to instructors.
  - ample *well-designed* remediation (not just repeating the same approach), enrichment and support to disadvantaged students.
  - accelerated language learning to catch up non-native speakers.

DEI advocates can at times be their own worst enemy by clinging onto the next misunderstood hope - most recently, Data Science as the “great equalizer.” Although we are in full agreement that Algebra II/Calculus should not be the only goal, *true* data science is not that much more lenient. It requires reasonably difficult mathematics coupled with *programming*. (How is that DEI-helpful?) A far better approach is outlined in 2.a.iv above.
Possible U.S. Phased Plan

A steady, long-term plan to shift mathematics education in the U.S. would comprise two phases, which must be overlapping in order to reach results in less than a decade:

**Throughout:** engage discreetly and increasingly with key positive jurisdictions (MA, MN, else?) and organizations (CBMS, etc.). Timing TBD.

**Track 1:** duration five years:
- Years 1-3: Design a deep Common Core-based curriculum\(^{39}\) and courseware\(^{40}\) after serious analysis of the macro/meso/micro aspects, while embedding core concepts and competencies.\(^{41}\) Socialize for acceptance.
- Years 3-5: Significantly expand (fund) teacher PD, particularly in DEI-sensitive areas, based on the above.

**Track 2:** duration eight years, starting alongside track 1.
- Years 1-3: Redesign K-12 Common Core standards to reflect more modern mathematics while making hard choices about essential content. Design three pathways reflecting “Produce/Interpret/Appreciate.”
- Years 1-3: work with college admissions to reform admission requirements. Significantly advertise the change to parents and teachers via a well-constructed communications plan.
- Years 3-6: Revisit the curriculum/courseware designed in 1) above and redeploy teacher PD in years 5-8.

Such a plan would require the project management skills of a change management consultancy, along with associated marketing/PR.

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\(^{39}\) No need to start from scratch, there are even free options like [https://jumpmath.org/us/](https://jumpmath.org/us/)

\(^{40}\) Yes, optional, but that’s a good example for many to follow, particularly in synch with the PD.

\(^{41}\) Which teachers *cannot* figure out on their own, for lack of time and expertise.
Appendices

Examples of Meso and Micro Problems of Students

Grade 7 Using https://new.assistments.org/

AssistMents facilitate teachers to provide their students with appropriate problems within an online learning environment. It is unique in that the selection of the problem for the students is entirely in the hands of the teacher: one significant advantage of this is that this removes the liability of students to game the system.

For example, if the student selects the problem(s) to be solved then they can choose the line of least resistance and choose ones that they know they can solve, rather than the ones that will support them to progress with their mathematical education.

AssistMents is a problem aggregator. The activities within AssistMents are linked to the common core: http://www.corestandards.org/Math/. The organization accesses and collates open-source mathematics problems from which teachers can select those they wish their students to solve. These problems come from a variety of sources, for example:

- Illustrative Mathematics: https://view.publitas.com/kendall-hunt-publishing-company/illustrative-mathematics-overview-brochure/page/1
- Open Up Resources: https://openupresources.org/
- Eureka Math (EngageNY): https://greatminds.org/eurekamath

The initial scan of problems within AssistMents focussed on problems from Illustrative Mathematics, Grade 7. The database contains the following for each problem within the set:

- Number of students who attempted the problem
- The percentage of correct responses
- The “favorite mistake” - the most common wrong answer
- The unit to which the problem is attached
- The type of activity
- The type of response.

The data set was trimmed to remove

- problems that were identified as cool-down, warm-up, or practice problems
- problems that required an open-ended response
- problems with zero attempts
A label of “difficult” was attached to the 20% of problems with the lowest value of “percentage correct” – 163 problems. Others were labeled as “not so difficult.” The cut-off point was a “percentage correct” value of 44.6%

<table>
<thead>
<tr>
<th>Units</th>
<th>Difficult</th>
<th>Not so difficult</th>
<th>Grand Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unit 1 - Scale Drawings</td>
<td>7</td>
<td>38</td>
<td>45</td>
</tr>
<tr>
<td>Unit 2 - Introducing Proportional Relationships</td>
<td>14</td>
<td>109</td>
<td>123</td>
</tr>
<tr>
<td>Unit 3 - Measuring Circles</td>
<td>13</td>
<td>21</td>
<td>34</td>
</tr>
<tr>
<td>Unit 4 - Proportional Relationships and Percentages</td>
<td>33</td>
<td>99</td>
<td>132</td>
</tr>
<tr>
<td>Unit 5 - Rational Number Arithmetic</td>
<td>34</td>
<td>152</td>
<td>186</td>
</tr>
<tr>
<td>Unit 6 - Expressions, Equations, and Inequalities</td>
<td>37</td>
<td>134</td>
<td>171</td>
</tr>
<tr>
<td>Unit 7 - Angles, Triangles, and Prisms</td>
<td>21</td>
<td>78</td>
<td>99</td>
</tr>
<tr>
<td>Unit 8 - Probability and Sampling</td>
<td>4</td>
<td>31</td>
<td>35</td>
</tr>
<tr>
<td>Grand Total</td>
<td>163</td>
<td>662</td>
<td>825</td>
</tr>
</tbody>
</table>

Analyzing the number of difficult answers for each unit identifies difficulties at the meso level. Three units contribute the greatest number of problems to this bottom percent. This indicates that there are difficulties at the meso levels for these three units.

- Proportional Relationships and Percentages (33)
- Rational Number Arithmetic (34)
- Expressions, Equations, and Inequalities (37)

Not all units have the same number of questions, and so the percentage of difficult questions is more important when deciding which units to investigate.

<table>
<thead>
<tr>
<th>Row Labels</th>
<th>Difficult</th>
<th>Not so difficult</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unit 1 - Scale Drawings</td>
<td>15.56%</td>
<td>84.44%</td>
</tr>
<tr>
<td>Unit 2 - Introducing Proportional Relationships</td>
<td>11.38%</td>
<td>88.62%</td>
</tr>
<tr>
<td>Unit 3 - Measuring Circles</td>
<td>38.24%</td>
<td>61.76%</td>
</tr>
<tr>
<td>Unit 4 - Proportional Relationships and Percentages</td>
<td>25.00%</td>
<td>75.00%</td>
</tr>
<tr>
<td>Unit 5 - Rational Number Arithmetic</td>
<td>18.28%</td>
<td>81.72%</td>
</tr>
<tr>
<td>Unit 6 - Expressions, Equations, and Inequalities</td>
<td>21.64%</td>
<td>78.36%</td>
</tr>
<tr>
<td>Unit 7 - Angles, Triangles, and Prisms</td>
<td>21.21%</td>
<td>78.79%</td>
</tr>
<tr>
<td>Unit 8 - Probability and Sampling</td>
<td>11.43%</td>
<td>88.57%</td>
</tr>
<tr>
<td>Grand Total</td>
<td>19.76%</td>
<td>80.24%</td>
</tr>
</tbody>
</table>
The two units of most interest are Unit 3, which has the highest percentage of wrong answers, but few questions in the unit, and Unit 4, which has the second highest percentage of wrong answers.

**Analysis of the difficulties within each identified meso level enables difficulties at the micro level to be identified.** The difficulties within each unit are linked to the Common Core standards within each unit and comparing the correct answer with the Common Wrong Answer (CWA) enables an exact identification of the difficulty that students are having. One could argue that the meso level is linked better to the Common Core standard: either way the micro level is identified when we compare the correct answer with the CWA.

**Unit 3 – Measuring Circles**

The difficult problems are linked to 7G4 of the Common Core Standard: Know the formulas for the area and circumference of a circle and use them to solve problems, give an informal derivation of the relationship between the circumference and area of a circle in the Common Core standards. The associated skills are problems involving areas and circumference of circles.

<table>
<thead>
<tr>
<th></th>
<th>Count</th>
</tr>
</thead>
<tbody>
<tr>
<td>Area of Circles</td>
<td>3</td>
</tr>
<tr>
<td>Circumference</td>
<td>10</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>13</strong></td>
</tr>
</tbody>
</table>

Comparing the CWA with the correct answer showed that students did not have difficulty with areas or circumference calculations. Most CWA arose because students rounded to a different accuracy than the expected answer (no accuracy was requested in the problems.) This is an issue with the software rather than the difficulty of the questions.

The third question revealed a lack of special awareness in two dimensions. This reflected the theme reported from the analysis of grade 6 data,
Students selected the correct shape but calculated the wrong answer. They found the area of one quadrant, rather the difference between a square and a semi-circle.

**Unit 4 – Proportional Reasoning and Percentages**

The difficult problems are linked to 7RP2 (Recognize and represent proportional relationships between quantities) and 7RP3 (Use proportional relationships to solve multistep ratio and percent Problems).

<table>
<thead>
<tr>
<th></th>
<th>Count</th>
</tr>
</thead>
<tbody>
<tr>
<td>Percent Word Problems (7RP3)</td>
<td>26</td>
</tr>
<tr>
<td>Recognizing Proportional Relationships (7RP2)</td>
<td>5</td>
</tr>
<tr>
<td>Represent proportional relationships in equation (7RP2)</td>
<td>2</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>33</strong></td>
</tr>
</tbody>
</table>

When comparing the CWA with the correct answers for these questions three themes were apparent. Two of these themes highlight a mathematical process which students have difficulty with.

1. Difficulty with equivalent ratios that involve fractions.

   Example 1: The difficulty occurs when deciding whether to multiple or divide by a fraction.
The correct answer is C, but the CWA was D, reflecting a difficulty in choosing whether to multiply or divide fractions.

Example 2: The difficulty occurs with equivalent ratios and fractions.

The problem that needs solving is $2 \frac{1}{2}: \frac{1}{3} = 100 : x$.

The CWA identifies the multiplier of 40 but does not complete the multiplication $40 \times \frac{1}{3} = \frac{130}{3}$.

2. Reverse percentages

The difficulty here is failing to correctly identify the base quantity for percentage calculations.

The correct answer is $234 / 0.9$. The CWA answer was $234 \times 1.1$. This reflects a lack of understanding that the base quantity is the original population of turtles. This difficulty was identified in several problems.

It is worth noting that the inherently difficult problems identified from analyzing data from AssistMents were also identified by the subject experts.

3. Understanding word problems.
In several instances students found the quantity required rather than the percentage. The difficulty here is a literacy issue rather than a mathematical one.
## Table of Jurisdictional Pathways

<table>
<thead>
<tr>
<th>Jurisdiction</th>
<th>K-5</th>
<th>6-9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>Destination</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Australia (Western Australia)</strong></td>
<td>Common to end of grade 10</td>
<td>ATAR Specialist</td>
<td>College STEM</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td></td>
<td></td>
<td>ATAR Methods</td>
<td>College STEM</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>ATAR Applications</td>
<td>College non STEM</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Essentials</td>
<td>Vocational</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Foundation</td>
<td>Numeracy qualification</td>
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<td></td>
<td></td>
</tr>
<tr>
<td><strong>Canada (Alberta)</strong></td>
<td>Common course</td>
<td>Mathematics 10-C</td>
<td>Mathematics 20-1</td>
<td>Mathematics 30-1</td>
<td>College STEM</td>
<td></td>
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<tr>
<td></td>
<td></td>
<td>Mathematics 20-2</td>
<td>Mathematics 30-2</td>
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<td>College non STEM</td>
<td></td>
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<tr>
<td></td>
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<td>Mathematics 20-3</td>
<td>Mathematics 30-3</td>
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<td>Vocational</td>
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<tr>
<td><strong>Eire</strong></td>
<td>Common course</td>
<td>Junior Certificate, Higher</td>
<td>Leaving certificate, Higher Mathematics</td>
<td>College STEM</td>
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<tr>
<td></td>
<td></td>
<td>Junior Certificate, Lower</td>
<td>Leaving certificate, Standard mathematics</td>
<td>College non STEM</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Leaving Certificate, Mathematics</td>
<td>Vocational</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>England</strong></td>
<td>Common course</td>
<td>GCSE Higher/Tier</td>
<td>Further Mathematics</td>
<td>A-level Further Mathematics</td>
<td>College advanced STEM</td>
<td></td>
</tr>
</tbody>
</table>

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Comparison of CCR Math vs Common Core

Commentary on the CCR/CCSS Crosswalk

Philosophy

As with literacy and K12 science curricula, the CCR standards aim to support and guide students through the study of applied mathematics while building a foundation for the later study of, primarily, the use of mathematics in various disciplines and secondarily, advanced mathematics. This includes additional standards on experimentation, exploring open-ended mathematical questions with more than one appropriate solution, and the use of mathematics to support storytelling and arguments.

The CCR standards were developed in partnership with the Australian Curriculum, Assessment and Reporting Authority (ACARA), so they begin with a set of preschool standards and continue up through 9th grade. In Australia, students have the option to select their own course of study in the final years of secondary school. CCR is currently working on appropriate mathematical standards for grades 10-12 that account for this common fissioning of educational tracks.

Bringing in Ideas Earlier

Because CCR is interested in building up conceptual understanding of mathematics, there is, very often, no need to wait for the algebraic formality that is required to introduce them traditionally. For instance, a young child can develop a sense for the probability of pulling a red ball out of a bag with five other black ones, without understanding the formalism, which will come later.

Proportional Reasoning: Division and Multiplication

CCR is focused on developing proportional reasoning, starting with halves in kindergarten. Following discussions with Cambridge Maths about the development of proportional reasoning, CCR also has a more significant focus on conceptual division throughout the first three years of school than the Common Core. The Cambridge Maths research also led to the decision to introduce partitioning of a collection together with partitioning of a whole as separate standards in Grade 2. Many of the early CCR standards in the proportional reasoning thread align to geometry standards in CCSS. We believe the underlying intent to develop proportional reasoning is present in geometry...
standards like CCSS.1.G.A.3. As this thread flows into the challenging concepts of fractions and decimals, we have chosen to generally place related standards into the “Arithmetic” heading as well as to introduce simple versions of division related to the preschool standards about sharing early in the progression.

CCR introduces multiplication and division in simple cases in Grade 1, building slowly over time, while CCSS brings in multiplication in Grade 2 and fractions and division in Grade 3. CCR standards attempt to keep the focus firmly on the development of proportional reasoning abilities by linking multiplication and division together, and connecting them to fractions, from the earliest stages of discussion. This means that the earlier CCR multiplication and division standards (c.f. CCR.M.ARI.2.B.C,F) are focused on solving problems rather than explicitly building symbolic representations (c.f. CCSS.3.OA.A.4).

**Algebraic Reasoning: Patterns**

CCR focuses on the study of patterns in algebraic reasoning starting in preschool, and introducing patterns related to place value in kindergarten. CCSS does introduce related pattern thinking, but not until Grade 3 in CCSS.3.OA.D.9, “Identify arithmetic patterns [...] and explain them using properties of operations.”

**Spatial Reasoning with Practical Applications**

CCR introduces coordinate graphing concepts in Grades K and 1 in the context of reading and navigating with maps, while CCSS does not introduce coordinate graphing until it appears in full Cartesian coordinate plane form in Grade 5 (CCSS.5.G.A.1 & 2). CCR also includes discussion of both rotational and linear symmetries, with extensions

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43 Partition circles and rectangles into two and four equal shares. Describe the whole as two of, or four of, the shares. Understand for these examples that decomposing into more equal shares creates smaller shares.

44 c.f. Multiplication problems: Given an authentic real-world problem solvable by multiplication and asked to solve and explain their answer, students can recognize multiplication, represent the solution visually or symbolically, explain reasoning, and interpret their results in context.

45 Determine the unknown whole number in a multiplication or division equation relating three whole numbers. For example, determine the unknown number that makes the equation true in each of the equations 8 x ? = 48, 5 = ? ÷ 3, 6 x 6 = ?.

46 1. Use a pair of perpendicular number lines, called axes, to define a coordinate system, with the intersection of the lines (the origin) arranged to coincide with the 0 on each line and a given point in the plane located by using an ordered pair of numbers, called its coordinates. [...]  
2. Represent real world and mathematical problems by graphing points in the first quadrant of the coordinate plane and interpret coordinate values of points in the context of the situation.
to 3D symmetries in Grades K with the concept of symmetry revisited in Grades 2 and 4. CCSS only discusses reflection across lines of symmetry in Grade 4.

While CCSS introduces angles, lines, rays, and the terminology of parallel and perpendicular in Grade 4 as the basis of continued study in geometry, CCR focuses on the concept of a right angle as it appears in the world starting in Grade 1 and puts angles into real-world contexts in Grade 2.

**Probabilistic Reasoning: likelihood**

CCR begins discussions of informal probability (e.g., likelihood) in kindergarten and reinforces these ideas in every single grade up through Grade 5 which dives deeply into fractions and percentages. At that point, it is possible to work more formally with probability. In contrast, the CCSS introduces the concept of likelihood in Grade 7.

**Differences of Focus**

CCR and CCSS focus on different approaches to similar topics. The first question on the table for CCR was, “What is mathematics for?” This was followed by, “What mathematics does everyone need to know?” If a group of mathematicians, mathematics educators, and users of mathematics could not come up with a reason why citizens, broadly, should know a particular topic of mathematics, we set ourselves a high bar to include it in the standards, or give it very little “airtime” during instruction. On the other hand, for mathematics that illustrate fundamental concepts, CCR worked to create standards to introduce the essential ideas early. This provides many opportunities for spiral review of the essential understandings at different levels of detail. The CCR standards also try to shift focus towards practical applications, real-world contexts, and approaching the same ideas from as many perspectives as possible to facilitate transfer from school mathematics to real-world mathematics. CCR standards take into account ubiquitous technology such as digital clocks, cashless money, calculating and graphing capabilities, and GPS. Standards need to take technological change into account before they mandate the use of specific examples to address underlying concepts.

**Arithmetic**

CCR does not have standards that specifically address telling time or counting money. Time and money are used as *examples* in the larger thread of “units.” There are also continuing sequences of standards, beginning in kindergarten, that place arithmetic in real-world contexts to build computational practice into practical applications. A guiding principle behind the development of the CCR standards was to focus on standards where there was a clear answer to the question, “Why should I know this?” that applies to significant fractions of society. Thus, *computation* involving analog time
and money are included as common examples where different unit systems are in play, but with the decline of cash transactions and analog clocks, the amount of time spent recognizing coins and reading analog clocks has dropped dramatically in our society. On the other hand, the ubiquity of GPS means that more and more people interact with maps frequently. CCR includes a thread across Geometry and Discrete & Computational Mathematics addressing reading, presenting, and interacting with maps and networks. The Common Core standards most related to this topic are early standards about directional language and then, from a mathematician’s perspective, advanced standards about relationships between matrices and networks and general applications of geometry.

Geometry

While terminology is helpful, the CCR standards attempt to focus on learning everyday terminology in recognition of the fact that unused terminology will quickly vanish from student’s vocabulary. Whenever practical, the standards can be achieved using student’s explanations of mathematical concepts in their own words. This also means that the CCR standards introduce initial versions of concepts quite early to start growing students’ awareness and allow reinforcing key concepts and vocabulary over longer periods of time. One example of this is the use of “acute” and “obtuse” along with “right” to describe angles in the CCSS while the introduction of angles in CCR uses “right” together with “greater than,” “equal to,” or “less than” three years earlier.

For instance, in kindergarten, there is a CCR standard about starting with a composite shape and finding simpler shapes that could be used to build it. In the Common Core, the equivalent standard asks students to start with simple shapes and determine if they can combine them to form a composite shape. Similarly, the CCR standard about classifying geometric shapes is the inverse of the CCSS version, requiring students to recognize families of shapes that share properties rather than trying to name a particular level of the classification. CCR does not explicitly require classification based on number of sides (or regularity) after CCR.M.GEO.2.A.47 The focus is on breaking all 2D shapes into rectangles and/or triangles.

Algebra

CCSS does include study of linear growth patterns at various points in the standards, but CCR introduces both linear and geometric growth patterns from Grade 1 that

47 Geometric possibilities: Given a satisfiable (but sometimes non-specific) set of geometric properties and asked to create a geometric figure, students can recognize when the answer is not unique and draw and/or build one or more geometric figures that satisfy the given properties.
continue into the proportional reasoning and exponential thinking required later. The contrast between linear growth and nonlinear growth is stressed early and often, as well as using linear vs. exponential to demonstrate the paradigm rather than linear vs. quadratic, given the importance of exponential phenomena in real life. This extends into the use of models in the upper grades.

CCR applies arithmetic to functions with the goal of understanding the effect on the graph of the resulting function, not so much on the pure arithmetic as CCSS.HSA.APR.D.7. CCR does not require rewriting equations into different forms only to reveal/explain properties of the function, but standards like CCR.M.ALG.8.F do require students to, “explain the impact of parameter choices on the graphs and the consistency of that impact across different types of graphs.” Similarly, CCR does not ask students to translate between recursive and explicit forms. The choice of which form of a relationship to use is covered in the thread of choosing an appropriate representation for the mathematics in your problem.

Algorithms

CCR focuses on the algorithm component of the standard algorithms. This includes algorithms for addition or subtraction, ordering using place value, multiplication or division, multiplication by fractions, and interpreting the order of operations. CCSS focuses on specific types of comparisons relating to fractions and place value, while the CCR standard is on a deeper level asking about what it means to order something and how to explain what you are thinking when ordering a set. Other than in the Standards for Mathematical Practice, which are rather generic, the CCSS has surprisingly few standards asking students to choose how to approach a problem and explain why they made their choice. As part of the algorithms thread, CCR has explicit standards about logical language. The CCR collection of standards on encryption are addressing the cluster of "experiment with transformations" in the CCSS standards in a different domain of mathematics.

Beyond the Common Core

Through re-focusing some ideas and minimizing or eliminating others, the CCR standards have created some space for including modern mathematical studies that are not addressed in the Common Core.

Practical Graphing, Statistics, and Probabilities

There are threads related to practical data representation in the CCR standards from preschool all the way through high school, including many ways to visually present data and questions to ask about whether a particular presentation of data is the best choice
for the situation. This includes bringing in Bayesian statistical reasoning in simple ways from Grade 1 and progressing to reasonably complex statistical reasoning by early high school.

CCR has additional focus on interpreting graphical data because of the ubiquity of graphical data presentations in everyday life. CCR mostly works from graphs or data to equations or interpretations, not from equations to interpretations directly. The assumption here is that equations are the intermediate step in all non-STEM fields. In situations where an equation is the starting point for a problem where graphing is helpful, we expect students to use appropriate technology (either paper/pencil or graphing technology) to produce a graph to analyze. CCR also requires students to work with multivariate data.

Discrete and Computational Mathematics

A variety of concepts from discrete and computational mathematics are included in the CCR standards that are not present at all (or at best in very different form) in the CCSS. Beginning in preschool, there is an ongoing thread addressing algorithmic thinking that starts with giving simple directions, moves into control-flow concepts like if-then branching, and continues into the concepts of formal mathematical proof using arithmetic. CCSS only addresses algorithmic thinking in the high school standards as “Building Functions” (e.g., CCSS.HSF.BF.A.1A) and through formal geometric proof. It is our experience that few students learn any transferable algorithmic thinking from the current studies of geometric proof. By switching the mathematical content where algorithmic reasoning is presented, CCR hopes to increase the number of students who can apply algorithmic thinking outside of the classroom.

CCR also includes standards from game theory such as CCR.M.DC.2.D on strategic guessing, or CCR.M.DC.4.D on narrowing the solution space, as well as expanding greatly upon the extra CCSS standard about analyzing decisions and strategies using probability concepts (CCSS.HSS.MD.B.7) into an 8th grade unit addressing 10 standards over 13% of the year.

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48 1a. Determine an explicit expression, a recursive process, or steps for calculation from a context.*

49 Strategic guessing: Given an ordered set of items or numbers in context (e.g., birth dates, guessing game) and asked to find a particular item or specific number, students can use a strategy to reduce the number of guesses needed to find a particular item in the set (e.g., binary search, dictionary).

50 Narrowing the solution space: Given a puzzle where efficient solving requires narrowing the solution space (e.g., coin weighing, guess my number) and asked to solve the puzzle, students can make moves or ask questions that incrementally narrow the solution space, rather than random moves or moves that affirm what is already known (e.g., "Is the number less than 20?" rather than "Is the number 20?").

51 (+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game).*
CCR includes standards from complex systems theory. CCSS uses geometry to present the concept of intricate relationships arising from a few simple axioms; CCR uses chaos theory and fractals. The closest Common Core standards are about interpreting and reasoning with equations, one about choosing a level of accuracy when reporting a measurement, as well as three standards about iterative and recursive processes. CCR has replaced these with 10 standards covering 28% of the year in Grade 9.

Timing

While the CCSS do not provide any direct guidance towards how much time should be spent on each standard, the CCR standards include an approximate percentage of total mathematical study each year that should be allocated to each standard. Even beyond the Given/Asked/Can structure of the CCR standards, this timing provides curriculum designers and instructors with another level of information about how deeply students are expected to engage with the material.

Standards for Mathematical Practice

The CCSS includes Standards for Mathematical Practice, a collection of eight practices, “that mathematics educators at all levels should seek to develop in their students.”

These Standards for Mathematical Practice do capture many aspects of how mathematicians think and behave.

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

The CCR approach looks at education at varying levels of detail to try to elucidate the complexities for teachers, by splitting “practices” into understandings and actions. To do this, CCR distilled the foundational knowledge structures, which we call Core Concepts, and the ability to apply, reflect and adapt, which we call Competencies.

52 http://www.corestandards.org/Math/Practice/
**Core Concepts**

The CCR standards design process for a discipline is centered around Core Concepts: the foundational knowledge in each discipline that captures the essential structures, connections, and modes of thought that are the reasons *why* each discipline exists. Because mathematics is a highly sequential subject, our work on identifying the Core Concepts of mathematics has also led to categorizing them at various levels of abstraction: discipline, branch, and subject/topic. For more detailed information on Core Concepts, please refer to CCR’s document “Mathematics for the Modern World” (section named: “Mathematical ways of thinking”) and the visualizations website.

**Competencies**

CCR has identified in its extensive research which competencies (a.k.a. “21st Century Skills”/Social-Emotional Learning (SEL)) are most applicable to Mathematics. Competencies may be mentioned at the Standards level but can only be activated via curriculum and instruction. The table below summarizes the CCR results:

<table>
<thead>
<tr>
<th>Discipline</th>
<th>Skills</th>
<th>Character</th>
<th>Meta-Learning</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Creativity</td>
<td>Critical thinking</td>
<td>Communication</td>
</tr>
<tr>
<td>Mathematics</td>
<td>Core</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

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## Goals & Functions of Mathematics

<table>
<thead>
<tr>
<th>K-12 Math functions</th>
<th>Thesis</th>
<th>Antithesis</th>
<th>Synthesis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Everybody needs these skills</td>
<td>Everybody needs these skills</td>
<td>Financial literacy, etc.</td>
<td>Teach Arithmetic (through proportional reasoning) more thoroughly</td>
</tr>
<tr>
<td>Teach Critical Thinking</td>
<td>Teach Critical Thinking</td>
<td>Citizens have to make decisions based on math (stats/prob really). We are even bombarded with stats and figures in our entertainment and from our peers!</td>
<td>Doesn't seem to really be happening with math as it is taught now. The people who learned problem solving attribute math as a cause, but most don't and they consider it a waste of time.</td>
</tr>
<tr>
<td>Teach Problem Solving</td>
<td>Teach Problem Solving</td>
<td>We all need to be able to formulate questions and think systematically and abstractly, it's more efficient and will help in all aspects of life. There are also more jobs that will demand this, as automatable things get automated.</td>
<td>Consider what we are actually asking students to do that we think will result in them learning problem solving for decision making. This involves a lot of translating world/language into math and math into the world/language, and teaching via visualization. (CC: Mathematizing)</td>
</tr>
<tr>
<td>K-12 Math functions</td>
<td>Thesis</td>
<td>Antithesis</td>
<td>Synthesis</td>
</tr>
<tr>
<td>---------------------</td>
<td>--------</td>
<td>------------</td>
<td>-----------</td>
</tr>
<tr>
<td>Training</td>
<td>Prepare Experts for STEM jobs</td>
<td>We are told there are huge skills gaps.</td>
<td>There might not be huge skills gaps (stats on this upon request), and even if there are, we shouldn't treat everyone like they're going to become mathematicians. Also it seems like many STEM jobs also use pretty basic math. Academic Math (&quot;Pure Math,&quot; proofs etc.) is essentially a different discipline from Math (Applied, fundamental/basic).</td>
</tr>
<tr>
<td>Employers to select employees</td>
<td>Proxy for Grit (After IQ) grit/conscientiousness is the most important determinant of success, and according to O*NET the most important quality in employees.</td>
<td>After adjusting for IQ and socioeconomic status, GPA measures grit perfectly fine. And we aren't even considering grit across different subjects. Maybe a brilliant artist just doesn't care about math and can't get himself to care when he sees no point, but would be super gritty with meaningful art assignments.</td>
<td>Not a good reason for math content, but should be integrated in competencies as resilience and meta-learning.</td>
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<td>Proxy for IQ/Cognitive Aptitude</td>
<td>For employers to select employees that will be successful at their jobs. Even in elementary school, can't be classified as Gifted/Talented unless your math scores are high.</td>
<td>IQ tests are taboo but we still use various methods to sort people by cognitive ability. Math is one of these ways. But there are much more direct and efficient ways (that also aren't IQ tests) that can do this job and not spend 1,700 hours of childhood on math.</td>
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<td>Weeding out process</td>
<td>Weeding out and then complaining that too many people are weeded out is not coherent and wasteful (to have our feet on the gas and the brake). We will need to figure out technological or other ways to sort through applicants.</td>
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<td>Political</td>
<td>Show off as a country (Optics)</td>
<td>The main impetus for math change seems to be comparisons with other countries.</td>
<td>Isn't it more embarrassing to fail? Also showing off is not a great reason for forcing kids to do something.</td>
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<td>Tradition</td>
<td>Learned automatically so why focus</td>
<td>Inertia, political courage</td>
<td>It is becoming more dangerous to not change, but that is hard to demonstrate until it is too late</td>
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## Use of Mathematics by Various Professions

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Examples of Occupations that Can Benefit from Modern Mathematics

For better viewing, see: https://curriculumredesign.org/wp-content/uploads/Modern-Mathematics-Usage-by-various-professions-CCR.xlsx