CENTER FOR CURRICULUM REDESIGN
Making Education *More* Relevant

Opinion Short Paper

Confusions in Mathematics

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In this short paper, CCR explores many of the tensions that exist in the design of effective Mathematics education. Although probably not a complete list, this paper highlights the dichotomies that are in play, reaching too often solutions that please no one.

Confusion between formality and importance (aka, "rigor")

What is considered beautiful in mathematics is the elegance and abstraction of certain structures and explanations. It is therefore natural to want to show students that beauty, and thus to teach them the formal language of mathematics. Unfortunately, learning abstractions is far more difficult than learning to reason with more concrete ideas,[1] *and the latter is much more useful in everyday life*.

For example, students often get confused by the formal notation involved in exponential expressions, but never learn about all of the real-world processes that follow the pattern of exponential growth, how to recognize graphs with this underlying structure, and how to interpret them correctly. Or, for example, students are not exposed to the idea of derivatives unless they pursue calculus, whereas the ideas behind calculus are accessible even to young children.[2] However, because they cannot access the formality, they are cut off from accessing the *concepts*, which we believe have value for enhancing thinking, even for those who never go on to implement them formally.

The framing of "**rigor**" has been an important element of this conversation. Everyone agrees that math classes should be rigorous, but there are different conceptions of what rigor in mathematics education ought to be. Often, the discussion focuses on traditional content, low pass rates for classes, and algebra.[3]
The material that seems more impressive, because it is more abstract and deeper in mathematics and out of the real world, is actually more *hollow* from a learning perspective. This is exacerbated by the

[1] https://en.wikipedia.org/wiki/Wason_selection_task
[2] Luba Vangelova (2014) 5-Year-Olds Can Learn Calculus. *The Atlantic*.
https://www.theatlantic.com/education/archive/2014/03/5-year-olds-can-learn-calculus/284124/
[3] What Is Rigor in Mathematics Really? *Dana Center Mathematics Pathways*
https://www.utdanacenter.org/sites/default/files/2019-02/what-is-rigor-in-mathematics.pdf

well-known dislike of applied math by theoretical mathematicians[4],[5]. Some research even suggests a relationship with the judeo-christian concepts of "purity"[6].
The sentiment of something being "not real math" when it is "too applicable" is a very unfortunate and powerful force shaping mathematics curricula around the world. When choosing between a standard that sounds simple but is complex logically not algebraically, and one that sounds impressive in its (usually algebraic) formality, the formal ones tend to win - ***it is a very common popular mistake to confuse algebraic complexity for "rigor***".

A more productive definition of rigor may be a "'deep, authentic command of mathematical concepts including procedural fluency and skills, conceptual understanding, and application."[7]

Confusion between coverage and mastery

As discussed above, coverage is not something easily defined. There are two effects that are of note. First, in typical discussions of standards, it is assumed that the ideal is for students to learn each standard to a level of mastery, so they can build on this knowledge. Of course, teachers cannot stop and wait for every student to achieve mastery of every standard before moving on. Therefore, although all the necessary topics get covered, *students who fall behind even a little will begin compounding their gaps, rather than filling them*. In fact, there are certain identifiable standards that are common places for students to get lost, and as a result, never be able to catch up again

The other aspect of coverage is that the focus becomes on demonstrating knowledge, which means that students can learn to parrot back on a superficial level what the teacher wants to hear, and it can be difficult to interrogate their understanding in a way that truly exposes the gaps. But a superficial understanding is not enough for transfer of the learning to the next standard that relies on mastery of this standard.

Confusion between discomfort of not knowing and actually doing something wrong

Human beings, like all living things, try to avoid discomfort. It is usually a signal that something is wrong, and avoiding it means we are doing everything right. However, sometimes growth calls for the

[4] Kary Mullis in Zeilberger, D. (1998). WZ Theory, Chapter II. arXiv preprint math/9811070
[5] exemplified by labeling Statistics as outside of "real" Mathematics with monikers such as "Mathematics & Statistics" or "Mathematical Sciences" - as if applied mathematics were not really mathematics…
[6] https://curriculumredesign.org/wp-content/uploads/Drift-towards-purity.pdf
[7] National Governors Association Center for Best Practices, Council of Chief State School Officers (2010) Key shifts in mathematics. As cited in What Is Rigor in Mathematics Really? *Dana Center Mathematics Pathways* https://www.utdanacenter.org/sites/default/files/2019-02/what-is-rigor-in-mathematics.pdf

exact opposite reaction: leaning *into* discomfort and examining its source. When math teachers feel discomfort with the material they are supposed to be teaching, they narrow their focus to concentrate only on what they are sure they know (or can copy from the materials), and avoid taking risks.

Students struggle with discomfort all the time, because learning itself is uncomfortable - it is taxing on cognitive resources. Becoming aware of a lack of some knowledge or ability is unpleasant, trying and failing is unpleasant, and putting in mental effort is unpleasant (or at the very least, tiring). This is perfectly natural, and it doesn't mean that students do not love learning. Just as climbing a mountain can be painful but worthwhile, many students can recognize when that is the case with their schoolwork - it is a question of cultivating a growth mindset, which Asian societies do well at in Mathematics[8] while Americans do well in sports. However, that doesn't change the discomfort they feel in learning.

Confusion between goals and assessment of goals

How do we know when a student has successfully learned something? What do they really need to learn, and what should they really just be exposed to? Standards do not tend to make these kinds of distinctions explicitly, leaving teachers to refer to exams and make those decisions for themselves. Assessment items, meant to score students' achievement, become the goals in themselves.

Meanwhile real goals, worthwhile aims that are multifaceted and applied, do not receive the attention they deserve because they are difficult to assess (precisely *because* they are worthwhile and complex). For example, many math curricula treat the mathematical idea of justification by teaching students formal proofs. If asked, math teachers say that they teach proofs to promote conceptual understanding, life-long mathematical skills and dispositions, to offer students an avenue for displaying their understanding, and so on.[9] And yet, too often, the assessments that target the content on justification do not test for these lofty goals, but rather, test the students' ability to manipulate mathematical formalisms.

The confusion comes into play when experts (those who understand mathematics) look at curriculum or assessment items and see the deep structures that they hope students will learn. Their curse of knowledge bias[10] prevents them from seeing how the majority of students, as novices, completely miss the underlying structure, and spend all of their energy focused on superficial features.

[8] https://www.oecd.org/pisa/PISA%202018%20Insights%20and%20Interpretations%20FINAL%20PDF.pdf
[9] Staples, M. E., Bartlo, J., & Thanheiser, E. (2012). Justification as a teaching and learning practice: Its (potential) multifaceted role in middle grades mathematics classrooms. *Journal of Mathematical Behavior*, *31*(4), 447–462. https://doi.org/10.1016/j.jmathb.2012.07.001
[10] https://en.wikipedia.org/wiki/Curse_of_knowledge

Confusion between teacher mental work and student mental work

A big part of a teacher's job is to break complex goals down into manageable pieces so that students can digest them. However, this can go too far[11]. As mentioned earlier, students *must* struggle to learn! So if a complex idea is broken down too finely, into a process that students can execute without worrying about the underlying concepts for example, they will most likely not absorb the deeper learning goals.

It is tempting to see one's students successfully executing the desired behaviors and conclude that they are learning, but if the teacher has done the work of mathematizing a scenario, breaking the problem down into pieces, and making the pieces resemble something the students have seen before, then the students have not had a chance to practice those things, and when it comes time for them to do it themselves, they will be unable to. This can be especially challenging to break out of, as it makes both students' and teachers' lives easier!

Confusion between length of list and bloat of curriculum

Worries about the amount of bloat of curriculum[12] have been around for a while and are well founded. After all, it is true that curriculum mirrors progress in the field (see Figure 1), and there are many standards which are included for historical reasons and have persisted due to inertia.

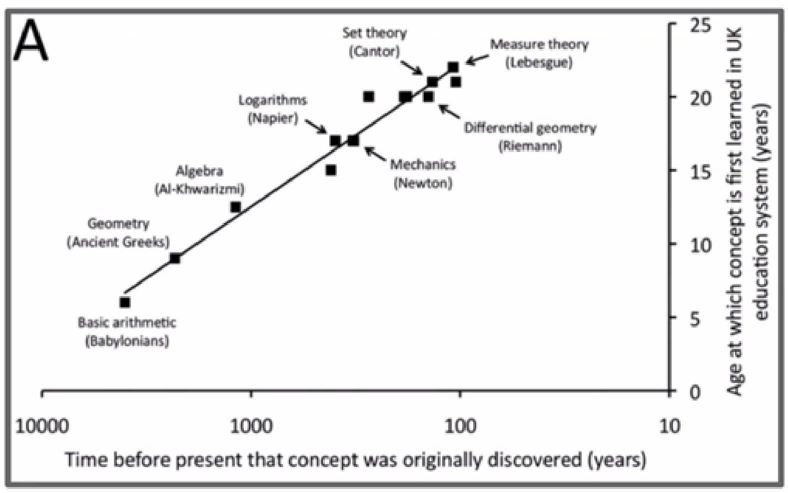

Figure 1: Education recapitulates discovery
Source: Mesoudi, 2011 [13]

[11] For a great compilation, see Cardone (2014) *Nix the Tricks*
[12] OECD (2020), Curriculum Overload: A Way Forward, OECD Publishing, Paris, https://doi.org/10.1787/3081ceca-en.
[13] Mesoudi, A. (2011). Variable cultural acquisition costs constrain cumulative cultural evolution. *PloS one*, *6*(3), e1

But the *number* of standards, while it may seem to serve as a reasonable proxy, **can be deeply misleading if it becomes the focus**. The result is that standards are not dropped, they are combined with other standards, making the resulting standards more confusing to navigate, but not decreasing the total amount of material students are being asked to learn.

Confusion by Politicians

It is the personal observation of one of the authors that most politicians/policymakers have been educated in the Humanities (often law) not STEM, and are therefore unable to challenge the academic mathematicians who threaten them with the collapse of their STEM competitiveness if their pet area of mathematics was to be lightened or removed (for instance, trigonometric functions - which have long been automated in theodolites).

Politicians feel discomfort with uncertainty, and organizations of architects and woodworkers will be a lot more vocal[14] in their attacks in support of trigonometry than the organizations of data scientists in support of making time and space available for data science.

Confusion by Parents

Parents feel discomfort when they do not recognize or understand what the purpose of their child's assignments are. It therefore becomes very difficult to try new content or approaches. Also, parents follow their own experiences, which in many cases was to recoil at the abstract structure of mathematics as taught to them. However, should university entrance requirements change, they will pivot, as "scorecards drive behaviors".

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[14] "…there is nothing… more dangerous to manage than a new system. For the initiator has the enmity of all who would profit by the preservation of the old institution and merely lukewarm defenders in those who gain by the new ones. " Nicolo Machiavelli in "The Prince"

Proliferation of (sometimes conflicting) Goals & Functions of Mathematics

K-12 Math functions:		Thesis	Antithesis	Synthesis
Everybody needs these skills	Everyday Math and Number Sense	Financial literacy, etc.	Very very limited actually needed. See Patton, J. R., Cronin, M. E., Bassett, D. S., & Koppel, A. E. (1997). A life skills approach to mathematics instruction: Preparing students with learning disabilities for the real-life math demands of adulthood. Journal of Learning Disabilities, 30, 178-187	Teach Arithmetic (through proportional reasoning) more thoroughly
	Teach Critical Thinking	Citizens have to make decisions based on math (stats/prob really). We are even bombarded with stats and figures in our entertainment and from our peers!	Doesn't seem to really be happening with math as it is taught now. The people who learned problem solving attribute math as a cause, but most don't and they consider it a waste of time.	Refocus on interpreting Math (especially Stats), not producing it. Refocus on the more or less basic kinds that actually appear in the news, not technical procedures (noisy data).
	Teach Problem Solving	We all need to be able to formulate questions and think systematically and abstractly, it's more efficient and will help in all aspects of life. There are also more jobs that will demand this, as automatable things get automated.		Consider what we are actually asking students to do that we think will result in them learning problem solving for decision making. This involves a lot of translating world/language into math and math into the world/language, and teaching via visualization. (CC: Mathematizing)
Training	Prepare Experts for STEM jobs	We are told there are huge skills gaps.	There might not be huge skills gaps (stats on this upon request), and even if there are, we shouldn't treat everyone like they're going to become mathematicians. Also it seems like many STEM jobs also use pretty basic math. Academic Math ("Pure Math", proofs etc.) is essentially a different discipline from Math (Applied,	Math that goes beyond what is needed for non-STEM jobs and being an informed citizen should be an elective like Art and Music. The fundamentals should be taught much more rigorously and the rest should be taught to give a basic understanding of its existence and usefulness.

			fundamental/basic).	Also math can be intertwined where relevant in other subjects (interdisciplinarity).
Employers to select employees	**Proxy for Grit**	(After IQ) grit/conscientiousness is the most important determiner of success, and according to ONET the most important quality in employees.	After adjusting for IQ and socioeconomic status, GPA measures grit perfectly fine. And we aren't even considering grit across different subjects. Maybe a brilliant artist just doesn't care about math and can't get himself to care when he sees no point, but would be super gritty with meaningful art assignments.	Not a good reason for math content, but should be integrated in competencies as resilience and meta-learning.
	Proxy for IQ/ Cognitive Aptitude	For employers to select employees that will be successful at their jobs. Even in elementary school, can't be classified as Gifted/Talented unless your math scores are high.	IQ tests are taboo but we still use various methods to sort people by cognitive ability. Math is one of these ways. But there are much more direct and efficient ways (that also aren't IQ tests) that can do this job and not spend 1,700 hours of childhood on math.	Not a good reason to keep math as a barrier. (High level math is required for everyone, including those (the majority) who only use low level math).
	Weeding out process		Weeding out and then complaining that too many people are weeded out is not coherent and wasteful (to have our feet on the gas *and* the brake). We will need to figure out technological or other ways to sort through applicants.	
Political	**Show off as a country (Optics)**	The main impetus for math change seems to be comparisons with other countries.	Isn't it more embarrassing to fail? Also showing off is not a great reason for forcing kids to do something.	Let's harness this for good, as wWe have with PISA
	Tradition	Learned automatically so why focus	Inertia, political courage	It is becoming more dangerous to not change, but that is hard to demonstrate until it is too late