CENTER FOR CURRICULUM REDESIGN
Making Education *More* Relevant

Core Concepts

Framing and Motivation

By Maya Bialik and Charles Fadel

"The goal [of teaching] should be not to implant in the students' mind every fact the teacher knows now; but rather to implant a ***way of thinking*** *that enables the student, in the future, to learn in one year what the teacher has learned in two years. Only in that way can we continue to advance from one generation to the next"* – Edwin T. Jaynes

Some Context:

There has been quite a bit of work on various forms of concepts, going by a variety of different names. A lot of this literature has separated out "knowing" from "doing" and "disciplinary" from "non-disciplinary"[1]. Many of them have also included some element of scale, although usually only two general levels[2] The following table illustrates how some of the most influential of these efforts fit together.

	Know		**Do**	
	Big Picture	Small Picture	Big Picture	Small Picture
Disciplinary	Big Ideas Essential Questions[3] Theories Principles Central Ideas Disciplinary Core Ideas Representative Ideas Threshold Concepts Enduring Understandings	Essential Questions[4] Microconcepts Generalizations	Core Tasks Processes Core Tasks Practices	Strategies Skills
Non-Disciplinary	Lines of Inquiry Essential Questions[5]	Macroconcepts Concepts Cross-Cutting Concepts	Transdisciplin-ary Skills	Subset Skills

Source: CCR: Summary table of types of terms used to describe Knowledge.
Understanding by Design (blue), Concepts Based Education (red), International Baccalaureate (orange), Next Generation Science Standards (teal), Realms of Meaning (purple), Meyer and Land (pink), and Rubicon (green)

[1] We use this term to include interdisciplinary, multidisciplinary, and transdisciplinary

[2] Which we generically label here "Big picture" and "Small picture"

[3] The type of essential question that deals with what is foundational: these point to big ideas and frontiers in discipline e.g. "How many dimensions are there in space-time?"

[4] The type of essential question that is necessary for deep dives into content e.g. "In what ways does light act wavelike?"

[5] The type of essential question that deals with what is timeless: these are interesting to debate and continue to change e.g. "What is justice"

The majority focus on *Disciplinary* "Big Picture" ideas, since that is central to the line of reasoning that focuses on the conceptual nature of knowledge. This is a starting point for a comprehensive, cohesive, and compact articulation of the most important concepts of each discipline, so that as curricula are designed and reviewed, this structure can fundamentally inform that process.

This paper will use Mathematics *as an example*, as per our experience, it is the discipline that conflates content and concepts the most; unlike, for instance, History, which, bowing to the impossibility to teach all of History, has managed to focus on Concepts in recent times.

Why are we choosing to highlight concepts?

Distinguishing concepts from content[6] allows us to consider and organize learning experiences more effectively. This should, in turn, enable us to take a more directed and informed approach toward selecting the optimal content for any given curriculum.

Approaches which blur the distinction between concepts and content are prone to falling into a coverage mindset "*in which students march through a textbook, page by page (or teachers through lecture notes) in a valiant attempt to traverse all the factual material within a prescribed time*", which leads to fragmented and inert knowledge[7], instead of usable, enriching knowledge.[8]

This also helps teachers, parents, and students:
- Teacher perspective: This framework helps me make sense of the content.
- Parent perspective: What is *really* important here for my child to understand?
- Student perspective: Why should I care? *What is the point*?

What does the research say about knowledge?

Research from the learning sciences shows that when people gain expertise[9] what they are really gaining is a more holistic way of viewing their discipline. Instead of fragmented details, they see concepts: patterns and deeper structures into which that information naturally fits.

These concepts are what allow us to deal with new information: we are constantly recruiting our understanding of old information to understand new information; this process is called transfer.[10] There are two well established[11] methods of transfer[12]:

- Low road transfer (a.k.a. reflexive transfer) - occurs more or less automatically. It

[6] Content in this case refers to what is left when concepts are pulled out of knowledge, in other words it is roughly equivalent to factual knowledge
[7] https://en.wikipedia.org/wiki/Inert_knowledge
[8] Wiggins, G., & Mctighe, J. (2005). Understanding by Design, Expanded 2nd Edition.
[9] Chi, M. T., Glaser, R., & Farr, M. J. (2014). *The nature of expertise*. Psychology Press.
[10] Hammer, D., Elby, A., Scherr, R. E., & Redish, E. F. (2004). Resources, framing, and transfer.
[11] Perkins, D. N., & Salomon, G. (1988). Teaching for transfer. *Educational leadership*, *46*(1), 22-32.
[12] Definitions adapted from: https://www1.udel.edu/dssep/transfer/definitions_of_transfer.htm

involves the triggering of well-practiced routines by stimulus conditions noticeably similar to those in the learning context.
- High road transfer (a.k.a. mindful transfer) - requires deliberate effortful abstraction and a search for connections.

The world outside of education will rarely present learners with carefully worded math problems or multiple-choice answers, nor the benefit of the book chapter as context. Individuals will need to have concepts internalized in order to recognize the deeper structure of similarity between their real-life situation and what they learned in school. For this reason, concepts are absolutely vital for students to learn.

Furthermore, in any discipline, the adage "use it or lose it" applies, due to the impermanence of content at high performance levels of a discipline, be it a human language, or mathematical language. However, Concepts endure: "Deceiving then explosive", "Small changes, big impact", "Possible is not probable", are memorable and relate to the real-world a student will encounter. Here is an example about Complex Systems and forest fires: if you are firefighters about to risk your life, don't you want to question your local government about whether the firebreaks and water wells have been placed strategically? Because you know that forest fires are fractals, and that "small changes, big impact". You want to make sure the policymakers and the mathematicians have done their job right (just as you can ask key questions to your accountant or lawyer).

Concepts? Content? What is the Difference? How can we separate them?

Typically, standards and curricula do not make a distinction between concepts and content, rather they tend to present them together. As a result, the exact difference may not be evident at first.
Put simply:
- *Content* is information. It is what is stored in encyclopedias and Wikipedia. Often it can be memorized.
- *Concepts* tie pieces of content together. They can be said to be the underlying currents or the overarching ideas of content. They are mental constructs that emerge from deeply internalizing a lot of content.

In other words, if the Content is the "what", then the Concept is the "so what?".

Concepts are what allow for high road transfer to take place. But of course, they must be carefully woven through educational experiences which are full of content knowledge. In fact, there is a symbiotic relationship between content and concepts, where each fits with and informs the other.

For Elementary school math, the concept of Comparison of quantities requires the learning of the content that can be described as ‘learning to use the symbols < and >, tied together with other content such as subtraction, ratio, relative difference, and percentages etc. Learning to use these symbols also contributes to students' learning of the concept of how Conventions in mathematics enable unambiguous communication of mathematical ideas and results.

When encountering more advanced content, students who have not internalized the concept behind comparison will be surprised that multiplying both sides by a negative number requires switching the direction of the comparison symbol, whereas those who have internalized the concept will find it natural, because they will understand the underlying reasons.

At this point it might be useful to mention that things such as “problem solving” live in a different part of our framework. Namely, Concepts are only referring to Knowledge, whereas problem solving is part of critical thinking, which is deeply related and will be carefully infused into the math curriculum. Notably, relevant knowledge is an important and underrated predictor of reasoning ability.

Are core concepts too advanced for children?

Concepts are refined and made more sophisticated over the course of an entire lifetime. Experts gain a thorough, precise, and formal understanding of concepts and this takes a lot of hard work and practice with technical details. The kernel of a concept, however, begins to be formulated (informally) far earlier. This allows further content to be connected to the idea, for transfer to occur, and for understanding to snowball and gradually become more and more sophisticated.

The most useful concepts for any given content area are worth articulating and designing learning experiences around. These we call Core Concepts, because they are fundamental constructs required to build a coherent picture of the content area at hand. If concepts are tools in our mental toolbelt, Core Concepts are the powertools.

For the kernel of a concept to be recognized early and thus be built on effectively, learning experiences must be designed with that goal in mind, and students’ attention must be intentionally directed to the concepts and their abstraction and articulation.

We are not saying that hard work is not required, just that exactly how much, what sort, in what context, and for what purpose, is a more complicated question. The goal is for regularities to be noticed such that core concepts can be abstracted, and the best way to achieve this will vary from person to person.

A Mathematician's Lament
by Paul Lockhart

A musician wakes from a terrible nightmare. In his dream he finds himself in a society where music education has been made mandatory. "We are helping our students become more competitive in an increasingly sound-filled world." Educators, school systems, and the state are put in charge of this vital project. Studies are commissioned, committees are formed, and decisions are made— all without the advice or participation of a single working musician or composer.

Since musicians are known to set down their ideas in the form of sheet music, these curious black dots and lines must constitute the "language of music." It is imperative that students become fluent in this language if they are to attain any degree of musical competence; indeed, it would be ludicrous to expect a child to sing a song or play an instrument without having a thorough grounding in music notation and theory. Playing and listening to music, let alone composing an original piece, are considered very advanced topics and are generally put off until college, and more often graduate school.

As for the primary and secondary schools, their mission is to train students to use this language— to jiggle symbols around according to a fixed set of rules: "Music class is where we take out our staff paper, our teacher puts some notes on the board, and we copy them or transpose them into a different key. We have to make sure to get the clefs and key signatures right, and our teacher is very picky about making sure we fill in our quarter-notes completely. One time we had a chromatic scale problem and I did it right, but the teacher gave me no credit because I had the stems pointing the wrong way."

Here we see a clear example, as math (often viewed in popular culture as procedural and dry) is mapped onto music (frequently thought of as intrinsically rewarding). Sure, playing a very complex piece of music with technical precision, or understanding the details of music theory is too lofty a goal for a beginner. But concepts can be conveyed in multiple informal and intuitive ways, and the important consideration is to be purposeful in connecting the work at hand to the overarching concepts and their applications in context.

The Specifics

Our formalization of Core Concepts takes what we have learned from others' work in this area, and builds on it. There are several things to note:

1. Each level of content contains its own core concepts (i.e., Discipline[13], Branch[14], Subject[15]) such that any given topic can touch on the core concepts associated with any of the levels that topic is contained in. (For example, when studying congruent and similar triangles, we may draw upon core concepts of geometry such as "scale", and upon core concepts of math such as "equivalence")
2. The concepts are not definitions. Though it is tempting to simply enumerate the most prevalent pieces of content and define them, we must resist such temptations. It is only through careful consideration of the cognitive utility of any potential concept

[13] E.g. Math, Science, Foreign Language
[14] E.g. Geometry, Arithmetic, Algebra
[15] E.g. Exponential Growth, etc.

formulation in relation to the content in its discipline/branch/subject that we can achieve productive insights. As a guideline, we have found it useful to make sure that the concepts are in the form of informative declarative sentences. (For example, in trying to pull out the concepts of Game Theory, an early draft included “the minimax algorithm” as the statement of the concept, but through reflection and iteration, this eventually became the concept “priorities affect strategy” and the minimax algorithm was added in a footnote as a great content piece to illustrate or connect to this core concept).

3. The concepts are not statements about importance. It is tempting to say X is very important, or X is important for Y. The challenge is to explain exactly why something is important. e.g. “different types of averages are useful for different things” vs. “use median when you don’t care about extremes - e.g. you don’t want to make a distinction between millionaires and billionaires”
4. Formatting:
   a. We have adopted the format of “Short Phrase: longer declarative sentence(s)” so as to be able to easily refer to a core concept while still distilling it down to a complete, informative sentence. [For example, **Equivalence**: Two (or more) mathematical representations can refer to the same mathematical object.]
   b. It is important to keep track of related content, since that is what everyone is used to tracking. We therefore often note relevant topics in parentheses after the “e.g.”. [For the previous example: (e.g. 3/6 and ½ via equivalence of fractions, or all 30-60-90 triangles via similarity, or 1 and 5 as representatives of the same equivalence class modulo 2)].
5. There are connections between concepts, but they are not the focus of our documents. In the long-term, as the framework is stored in databases, all of the interconnections can be tracked in a more convenient format.
6. Concepts need to be taught both top-down and bottom-up. Concepts are the cognitive tools that come with a deep understanding of different pieces of content. That said, simply reading a list of them will not be enough to result in high-road transfer. It is the combination of the (well-designed) bottom-up educational experiences of students and the top-down emphasizing of core concepts that will lead to successful abstraction and internalization, and eventually to high-road transfer. This comes in part by having the “so what” articulated at the top level in ways that inform and empower the practitioners who are curriculum designer-implementers in their own right.

As a summary:

- **Core Concepts transfer to real life**
- **They are conflated by Maths as Content, and therefore under-taught**
- **They can be taught without algebraic manipulation, therefore:**
  - **They can be taught in earlier years**
  - **They can be taught to non-STEM HS students**
- **They MUST be understood by ALL students.**

APPENDIX

Discipline-level and Branch-Level Math Core Concepts

By Emma Smith-Zbarsky and Janet Slesinski

Discipline-Level Core Concepts (DCCs) - 8

Equality and Relationships - "Same thing, new look"
The same mathematical object can be represented in different ways. Operations that preserve equality can be used to manipulate one representation of an object into a different representation that may be more informative in a given context. We can also use mathematics to determine whether two representations are equivalent. Some objects and representations are not equivalent, but still have meaningful relationships that can be defined mathematically, such as inequalities, functions, or symmetries.

Patterns and Structure - "I've seen this before"
Patterns and structures allow us to generalize and create consistent systems for organizing information. We can recognize, create, and communicate patterns, symmetries, and relationships both visually and symbolically. We can choose the right abstraction to represent entire classes of mathematical structures at once, leading to new insights. We must state the assumptions that are necessary to communicate the structures we have created.

Systematic Approach - "Split problems into manageable chunks"
When approaching a problem we need to determine what we know as well as what we want to know and recognize potential paths to explore to eventually connect the two. We can use logical reasoning to find the best path and to convince people of a conclusion. When faced with challenging problems we can divide the problem into more manageable pieces, keeping in mind we may revisit our choices at any time in the process. Finally, we should consider the validity of intermediate and final results in different situations.

Numbers and Quantities- "Numbers are everywhere"
Humans invented numbers to count things, and realized they could be used for so much more. Numbers surround us and we must recognize, relate, connect, and appreciate them in their many forms. Familiarity and flexibility with numbers and the operations we can perform with them allow us to measure, approximate, estimate, and anticipate solutions to problems before we begin solving them.

Spatial Reasoning - **"A picture is worth a thousand words"**
We represent information spatially to allow our ability to reason about space to enhance our intuition and computational abilities. Visual approaches to problem solving allow us to process abstract ideas in a more concrete way.

Modeling - **"All models are wrong, some models are useful"**
Given a real-life scenario, we often have to build a model that is imprecise in order to make predictions. In some cases, information may be available that is irrelevant to the particular question we are trying to answer and should be ignored when formulating the mathematical statements involved in trying to solve the immediate problem. By carefully tracking simplifying assumptions made, we can use imprecise models to make estimations and draw conclusions about our scenario. We consider the constraints of the model to determine the implications of

our results and decide whether we must modify our model and repeat the modeling process to reach a valuable solution.

Conventions - **"Math is a language too"**
To communicate mathematics, we use mathematical language, which has its own terminology, notation, symbols, graphs, diagrams, shorthand, and vocabulary.

Simplicity and Elegance - **"Everything should be made as simple as possible, but not simpler. (A. Einstein)"**
There is frequently more than one way to solve a problem or represent a pattern, but they may not be equally efficient or understandable. The strategies that you choose should be based on your priorities - maximizing payoff and/or minimizing risk or cost. A solution is elegant if it elucidates deep structure in a way that makes that solution easier to reconstruct.

Branch-Level Core Concepts (BCCs) - 27

Arithmetic - 4

Proportional Reasoning - "Rescaling for a purpose"
In additive thinking, we work in terms of absolute differences. In multiplicative thinking, we change our perspective or scale and think in terms of constant ratios. Proportional reasoning expands from additive and multiplicative thinking to include exponential thinking and applications of scale factors, fractions, ratios, rates, and proportions.

Comparison - "How do things measure up?"
We can describe how mathematical objects relate to each other, even if they are not equivalent, if they have some comparable qualities. By applying our understanding of place value, we can estimate or compare values or operate practically with very large or very small numbers.

Decomposing Numbers - "Numbers are made up of one another"
We can express numbers in terms of other numbers and we can use this to our advantage when manipulating them. We can decompose by looking for bridges to 10 (such as 6+4 or 3+7), place value (such as 32 = 30 +2) or by factoring (such as 60 = 15*4). We can decompose numbers as abstract patterns such as a^2-b^2=(a+b)(a-b) that can allow us to quickly reformulate relationships. We can use decomposition to classify numbers such as odd/even or prime/composite.

Bases - "Not always base 10!"
Numbers are symbols that represent groups of a power of a particular base. In the conventional place value system, each digit represents a group of a power of 10, but this could be a different number in a different base system. Other common bases include 2 and 16.

Algebra - 5

Abstraction - "x is for unknown"
On the smallest scale, we apply abstraction when we create variables to represent unknowns, express relationships generally, as well as to represent desired solutions during mathematical computations. A variable may represent a potentially infinite set of values. When grouping components by choosing an aspect and ignoring other aspects, it is possible to describe entire classes of mathematical structures at once. Careful analysis of such abstractions can allow standardized approaches in concrete problems.

Functional Thinking - “Patterns are powerful”
Variables can be considered independent or dependent and be used together to create a rule that represents a function. Rules can associate one set of mathematical objects with another. Identifying regularities allows us to make generalizations, thereby helping us to recognize and describe principles of how things behave. By observing patterns, we can generate additional elements in a pattern or communicate the pattern.

Coordinate System - “Grids communicate space”
The location of objects or a position in space can be communicated through the use of a coordinate system. When placing a coordinate system, we have the freedom to choose our own orientation, types of coordinates, choice of origin, etc., each with its own set of assumptions.

Inverses - “the undo button”
Some operations or functions have an inverse operation or function which undoes the effect of the operation or function. Pairing an operation or function with its inverse within the appropriate mathematical system creates an identity.

Unitizing - “comparing apples to apples”
Mathematizing requires consistency. To express a value, we define a unit that allows us to keep track of any value in terms of this consistent unit. The idea of combining like terms is a special case of adding elements of the same unit.

Geometry - 2

Dimensions - “Unraveling complexity with dimensions”
Dimensions provide a frame of reference for defining space and size. They are a fundamental measure of complexity and allow tracking of multiple features at once. Often dimensions can be collapsed or projected down to lower the complexity of the problem by ignoring certain features to enable focus on others. Considering the boundaries where a set or space of a higher dimension ends can allow us to view the components of lower dimension that are embedded inside the set or space.

Geometric Transformation - “Preserving and Altering Properties”
Mathematical objects can be systematically re-expressed, keeping certain properties while changing others. When an object changes its location, orientation, shape, or size, some attributes remain the same and some may change.

Statistics - 4

Variation - “There is no such thing as perfectly average ”
Most events in the world are not perfectly predictable, meaning that there is some level of variability in the outcomes. Statistical relationships determine how well one or more input variables can explain the variability of an output variable. By studying the variation in a sample we can try to untangle randomness from correlation or causation. The purpose of statistical methods based on samples is to form judgments about the parameters of a population and the reliability of statistical relationships.

Asking answerable questions - “Ask the right question, to get the right answer”
We need to formulate questions in ways that inform data collection, and consider what questions can or cannot be answered by referencing or investigating given data. We need to

consider what we want to be able to compare, how to set it up, and how to treat all the related factors. Proper experimental design allows us to distinguish between correlation and causation.

Gather data - "garbage in/garbage out"
Data comes in many forms from categorical to continuous and depending on its form and quantity needs to be measured and analyzed differently. Whether we gather the data ourselves or use existing data, we use a sample to identify a subset of a population of interest so that insights gleaned from the sample can be applied to the population. We estimate how much data is necessary to answer our question. The method by which a sample is identified may introduce biases which result in the sample doing a poor job of reflecting the population of interest.

Visualizing and Summarizing Data - "choose the right tool"
Visual and numerical summaries of a dataset allow us to identify shapes and patterns and make decisions in ways the full dataset often does not. We should choose the right representations for the given data set and type of data to highlight different features and relationships among features. Because creating and choosing summaries can be equal parts science and art, care is necessary when interpreting summarized data.

Probability - 3

Uncertainty - "Possible does not mean probable"
Probability allows us to quantify uncertainty in our everyday lives. If we can count all the possible outcomes in a sample space and the number of outcomes of interest, we can quantify exactly how likely something is to happen by finding the probability. If we can't count the entire sample space, we can consider uncertainty as a degree of belief and update it as we acquire more information.

Law of Large Numbers - "probabilities work out as expected, but in the long run"
When an experiment is repeated many times, the average of the results will tend to stabilize around the expected value. The more trials are carried out, the more the observed proportion of events begins to resemble their "true" probabilities. This long-term stability despite short-term variability is at the heart of many examples of good and bad decision making.

Combinatorics - "Counting possibilities"
Potential events, outcomes, networks and algorithms can be systematically explored through tables, probability trees, and other organizing structures and visualizations which help to break down complex problems. We often need to count the number of different ways a sequence of choices can be made, or the number of different ways of selecting or rearranging sets of objects.

Computational - 5

Solving classes of problems - "Laziness is good evolutionary design; why go one by one?"
A single algorithm, such as the quadratic formula, may be able to solve a large number of problems, even though it might not be the most efficient choice. However, there exist some important problems for which no efficient and optimal algorithm is known, and which may be unsolvable. Even when solving the problem may be difficult or impossible, checking a potential solution may be extremely easy.

Unambiguous Description - "Precise pathways solve problems"
The description of an algorithm must be precise enough that every step is specified with no room for ambiguity. A given algorithm can be described in a variety of ways, with different levels of formality. The usefulness of an algorithm depends on several factors, such as whether it always produces correct solutions, whether it terminates, and whether it can be completed with a reasonable amount of resources.

Exploring Algorithmic Bounds - "check edge cases"
One systematic method of exploring algorithmic bounds is to look for edge cases that form the boundaries in some configuration space. It is often difficult to recognize situations that may "break" an algorithm such as unrecognized division by zero. A rigorous check for accuracy requires accounting for all the edge cases.

Accurate Enough? - "don't be pseudo-precise!"
Bounding the error between the output of an algorithm and the exact solution to a problem allows selection of "good enough" algorithmic solutions. Heuristic algorithms allow quick estimates of a value. Other algorithms, such as long division, may be arbitrarily accurate depending on how many iterations are included.

Algorithmic Simplicity - "Best method, yes, but for whom?"
We can compare algorithms based on how complicated they are as measured by elegance, speed, resource use, precision, in order to identify the algorithm that best matches our current needs. Different algorithms may be more efficient depending on the tools available and the preferences of the person. How one person solves a problem may not be how a computer solves a problem, or even how another person solves a problem.

Discrete - 4

Simulation Allows for Experimentation - "simulate but verify"
Even when a system seems simple, making good predictions can be challenging. A useful method to study many systems, regardless of their apparent simplicity or complexity, is to use a computer program to run simulations to see the larger scale patterns. This allows us to experiment with ideas and verify predicted results.

Sensitivity - "Small change, big impact"
In a complex system, a small change to a single parameter can have a big impact on the future state of the system. Despite the appearance of randomness, there may be patterns or chaos in the outcomes; some parameters can have predictable big picture impacts.

Elements Interact - "feedback is hard to predict"
Elements of a system can interact. This can lead to feedback loops, where an outcome of a system becomes an input to the same system. Such feedback can result in one or more equilibrium states, periodic behavior, or a value growing without bound or decreasing to zero. Even a system evolving according to simple, local rules can exhibit complex behavior at higher scales without the need for any external or top-down influence.

Playing the Game - "Play the right game right"
Many situations can be compared to game play, and in these situations, there is value in recognizing the game and simulating playing it out using the optimal strategies given our objectives. Not all games have a winner and a loser—in some games everyone can win, and in others, no one can.