The Mathematics People Really Need

Keith Devlin
Stanford University
A key observation

For several thousand years:

Calculation was the price we had to pay to learn and do mathematics.

That is no longer the case.
Why so much longer?

Took 140 years to build

Took 14 years to build
Why so much longer?

- It still requires good architecture, design, and sound construction
- The basic principles are the same
- What changed were the tools available
- Today’s architects, designers, and builders need different skills than their forebears of earlier generations

Took 140 years to build

Took 14 years to build
Some of today’s mathematical tools

- Google
- Wikipedia: The Free Encyclopedia
- Gmail
- WolframAlpha
- YouTube
- MathWorks® MATLAB
- LinkedIn
- mathoverflow
- Wolfram Mathematica
- Quora
- Twitter
- Graphing calculator
- Notebook
Does this mean we no longer need to teach calculation?

✦ Of course not!
✦ But we no longer need to teach for accurate or fast **execution**
✦ Today, we need to teach for **understanding**
✦ Today’s users of mathematics need **different skills**
If you are connected with the world of K-12 mathematics education, it's highly unlikely that a day will go by without you uttering, writing, hearing, or reading the term "number sense". In contrast, everyone else on the planet would be hard pressed to describe what it is. Though entering the term into Google will return close to 38 million hits, it has yet to enter the world's collective consciousness. Stanford mathematician Keith Devlin explains what it is.
What now?

What and how do we teach for life in such a world?

We need to go back and reassess:

✦ Why mathematics is developed
✦ How it is developed
✦ How it is used (in the world)
How mathematics helps us understand the world and do things in the world

1. Real world situation
2. Technical description
   - Abstract & express in technical language
3. Mathematical description
   - Abstract & express in mathematical language
4. Mathematical results
   - Feedback
   - Do math
How mathematics helps us understand the world and do things in the world

- Real world situation
  - Abstract & express in technical language
- Technical description
  - Abstract & express in mathematical language
- Mathematical description
- Mathematical results
  - Feedback
  - Do math

Mathematical model
How mathematics helps us understand the world and do things in the world

- real world situation
  - abstract & express in technical language
  - feedback

- technical description
  - abstract & express in technical language
  - do math

- mathematical description
  - abstract & express in mathematical language

- mathematical results

Mathematical model
The mathematical method

- Identify a particular pattern in the world.
- Study it.
- Develop a notation to describe it.
- Use that notation to further the study.

- Formulate basic assumptions (axioms) to capture the fundamental properties of the abstracted pattern.
- Study the abstracted pattern, establishing truth by means of rigorous proofs from the axioms.
- Develop procedures that you and others may use to apply the results of the study to the world.
- Apply the results to the world.

Mainstream Mathematics

Axioms
Proofs
Procedures
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FOR THE MOST PART, CLOUD RESOURCES CAN “DO THE MATH” FOR US. WHAT WE NEED TO KNOW IS HOW TO USE THOSE RESOURCES!
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Mathematical Thinking

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**WE SOMETIMES FORMULATE UNDERLYING ASSUMPTIONS TO UNDERSTAND HOW WE ARE APPROACHING THE PROBLEM.**

**WE RARELY PROVE THEOREMS.**

**MOST OF THE TIME, WE USE TECHNOLOGY PRODUCTS TO EXECUTE MATHEMATICAL PROCEDURES, SOLVE EQUATIONS, DRAW GRAPHS, ETC.**
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AND OTHER TOOLS
Yesterday’s mathematics user

Had to be able to play many instruments
Today’s mathematics user

Has to be able to conduct an orchestra
What do you need to conduct the mathematical orchestra?

✦ Number sense (Gersten & Chard, 2001)
  ✷ Fluidity and flexibility with numbers: having a sense of what numbers mean, understand their relationship to one another, able to perform mental mathematics, understand symbolic representations, and can use numbers in real world situations.

✦ Deductive reasoning skills

✦ Creative problem solving ability
  ✷ algorithmic reasoning
  ✷ metacognitive skills
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So is content not important?

- Of course it is!
- The choice of topics can make a huge difference in terms of relevance to life today, (hence) to student engagement, and to educational efficiency.
- For instance, probability, statistical theory, and network theory are far better than calculus.
- Ultimately, what is really important, however, is how the topic is approached.
- Take geometry as an example:
Geometry – an ancient mathematical model

\[ BC^2 = AB^2 + AC^2 - 2AB \cdot AC \cdot \cos A \]

\[ r = \frac{2S}{a + b + c} \]

\[ S = \frac{a + b + c}{2} \cdot h \]

\[ S = \frac{1}{2} \cdot a \cdot r + \frac{1}{2} \cdot b \cdot r + \frac{1}{2} \cdot c \cdot r \]

\[ S = \frac{1}{2} \cdot ab \cdot \sin \gamma \]

\[ \angle ABC = \frac{1}{2} \cdot \angle AOC \]
But geometry is **much** more than Euclid’s theory.

As the mathematics of **shape**, geometry is about every aspect of our lives.

In today’s world, there are many more relevant shapes than triangles.

The real power of mathematics lies in its capacity to reveal **invisible patterns**.
Geometry is about visualizing our world and aspects of our life.

These are both Euclidean geometry:
But done in a way that is much more relevant today

Tony Robbin: using geometry to understand the complexity of life in a multi-cultural, multi-ethnic, multi-linguistic modern city
Robbin’s work was “art”; this is national defense

A geometric representation of the “invisible” structures of a modern battlefield

Capturing commands, information networks, communications, threats, plans, and capabilities

Interactive

Dynamically linked to the aerial view and any local ground views

How today’s pros use mathematical thinking

Heuristics
Heuristics

A heuristic technique (/hjuːrɪstɪk/; Ancient Greek: εὑρίσκω, “find” or “discover”), often called simply a heuristic, is any approach to problem solving, learning, or discovery that employs a practical method not guaranteed to be optimal or perfect, but sufficient for the immediate goals. Where finding an optimal solution is impossible or impractical, heuristic methods can be used to speed up the process of finding a satisfactory solution. Heuristics can be mental shortcuts that ease the cognitive load of making a decision. Examples of this method include using a rule of thumb, an educated guess, an intuitive judgment, guesstimate, stereotyping, profiling, or common sense.
# Mathematical heuristics

**PROBLEM:** A bat and a ball cost $1.10. The bat costs $1 more than the ball. How much does the ball cost on its own? (There is no special pricing deal.)

## SOLUTION 1

<table>
<thead>
<tr>
<th>Need to subtract</th>
<th>Have data $1.10 and $1</th>
<th>Subtract $1 from $1.10</th>
<th>Answer 10¢</th>
</tr>
</thead>
</table>

Heuristic
- Fast
- Usually works

## SOLUTION 2

Let $x =$ cost of bat
Let $y =$ cost of ball

$x + y = 1.10$
$x = y + 1$

Eliminate $x$: $1.10 - y = y + 1$
$0.10 = 2y; \ y = 5¢$

## SOLUTION 3

<table>
<thead>
<tr>
<th>COST OF BALL</th>
<th>COST OF BAT</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cost</td>
<td>$1 more</td>
</tr>
</tbody>
</table>

BALL + BAT = $1.10
BALL = 5¢

Heuristic
- Fast
- Always works

Procedure
- Slow
- Always works
**Mathematical heuristics**

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<table>
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<tr>
<th>Pre-rigorous thinking</th>
<th>Rigorous thinking</th>
<th>Post-rigorous thinking</th>
</tr>
</thead>
<tbody>
<tr>
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<td><strong>COST OF BALL</strong></td>
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Need to subtract
Have data $1.10 and $1
Subtract $1 from $1.10
Answer 10¢

Heuristic
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Usually works

Rigorous thinking

Let $x = \text{cost of bat}$
Let $y = \text{cost of ball}$

$x + y = 1.10$
$x = y + 1$

Eliminate $x$:

$1.10 - y = y + 1$

$0.10 = 2y$

$y = 5¢$

Heuristic
Fast
Always works

Post-rigorous thinking

COST OF BALL
Cost

COST OF BAT
Cost $1 more

BALL + BAT = $1.10
BALL = 5¢

Heuristic
Fast
Always works
How to develop post-rigorous thinking

- Almost certainly, we need to go through rigorous thinking. **Lots of practice** is required for the development of a good set of post-rigorous heuristics.

- But the standard algorithms and procedures were **optimized** for efficient (human) **performance**. Computer systems now handle all of that.

- So, we should teach algorithms and procedures **optimized** for **understanding**, so people can make effective, safe use of those computer systems.

- With unlimited, fast computer power and easy access to masses of data, learning can be achieved by working on **meaningful**, real problems with **real data**. (Project-based learning. Teams)

- **Content is no longer primary**. Focus should be on the **way of thinking**.
Further reading

http://devlinsangle.blogspot.com/2018/04/

How today’s pros solve math problems: Part 3 (The Nueva School course)

By Keith Devlin

You can follow me on Twitter @profkeithdevlin

NOTE: This article is the final installment of a four-episode mini-series posted here starting in mid-January. In writing it, I have assumed my readers have read those three earlier pieces.

At the end of last month’s post, I left readers with a (seemingly) simple arithmetic problem. I prefaced the problem with the following two instructions:

1. Solve it as quickly as you can, in your head if possible. Let your mind jump to the answer.

2. Then, and only then, reflect on your answer, and how you got it.

The goal here, I said, is not to get the right answer, though a great many of you will. Rather, the issue is how our minds work, and how can we make our thinking more effective in a world where machines execute all the mathematical procedures for us?