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Mathematics for the 21st Century

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Does Mathematics education enhance Higher-Order Thinking Skills?

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Introduction

"Let No One Ignorant Of Geometry Enter Here"

This phrase, reportedly engraved at the door of Plato's Academy[1], conveys the ancient idea that is still fueling our lives today: mathematics is a prerequisite to philosophy and therefore access to wisdom and truth. The idea that mathematics is not only practically useful and theoretically beautiful but also broadly intellectually and cognitively beneficial is still predominant today, justifying its fundamental role in global educational curricula and widespread importance in testing. Fig. 1 shows these three levels of benefit to the individual, progressing from practical benefits, to the higher-order cognitive benefits, and finally to the emotional benefits such as beauty.

To assess the legitimacy of the claim of the middle level -- the cognitive benefits of mathematics -- which is often cited as the justification for its central place in the curriculum, we will illustrate an historical overview of the beliefs about the cognitive benefits of mathematics, present research in psychology on thinking and transfer of cognitive skills across domains and finally, we will discuss findings from neuroscience on the development of mathematics and its influence on the brain. We will then re-examine the three tiered claim of the benefits of a mathematics education.

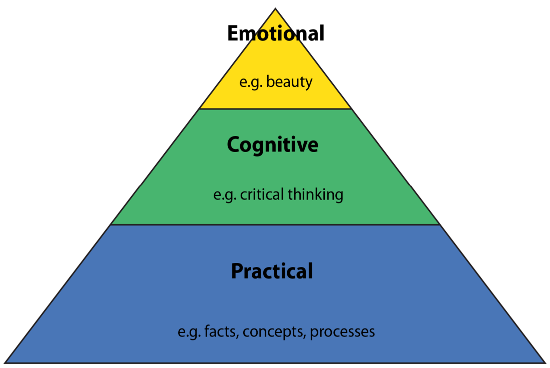

Figure 1: The three levels of benefit used as justification of the mathematics curriculum
Source: Charles Fadel, CCR

Historical overview

The conception of mathematics as a means to improve reasoning can be traced back to ancient Greece as Plato theorized in the seventh book of the *Republic*. Higher education for the guardians of an ideal society was to start with arithmetic, as it "leads to the apprehension of truth."[2]

> Arithmetic is a kind of knowledge which legislation may fitly prescribe; and we must endeavor to persuade those who are to be the principal men of our State to go and learn arithmetic, not as amateurs, but they must carry on the study until they see the nature of numbers with the mind only; nor again, like merchants or retail-traders, with a view to buying or selling, but for the sake of their military use, and of the soul herself; and because this will be the easiest way for her to pass from becoming to truth and being.

[1] For an overview of the literature on the topic, see http:// plato-dialogues.org/faq/faq009.htm
[2] B. Jowett, Ed., (1975) *The Dialogues of Plato.* Oxford Univ. Press, Oxford, p. 785.

Plato describes the purpose of learning arithmetic as reaching far beyond its practical applications to merchants and retail-traders, but "for the soul herself." At the same time, Plato does not consider the learning and practice of mathematics[3] to be reserved exclusively for the guardian class: "Even the dull, if they have had an arithmetical training, although they may derive no other advantage from it, always become much quicker than they would otherwise have been"[4]. This ancient view held that students would continue their study of mathematics with geometry, since it was seen as a prerequisite to develop the power of abstraction -- the ability to go beyond the level of sensible experience[5].

The modern view of the educational value of mathematics came from post-revolutionary France of the late eighteenth century where, based on the conviction from the Enlightenment that man was a rational animal[6] (first theorized by Aristotle in his Metaphysics and later on completed by Descartes in his *Discours de la méthode*), mathematics was taught as a way to develop this rationality. While this conception of mathematics gradually disappeared from French curriculum, mathematics remained nonetheless a strong selection tool in all the French elite higher education system, as a mark of status and a tool of social differentiation.

In England, the discussion about Mathematics in the curriculum was focused in Cambridge University, where the goal was to educate the clergy of the Anglican church. Until the 1860s, in order to receive a degree it was necessary to pass a grueling mathematics exam called the Mathematical Tripos. Opposition to this focus on mathematics came from Oxford university in the 1830s; the argument came down to whether reason was best taught through the rules of logic on their own or infused in mathematics. Although the discussion that grew from this was powerful and caused many to question the foundational assumption of the relationship between mathematics and reason, mathematics remained a stronghold in the curriculum. In 1906 J.D. Fitch, a contemporary educator, echoed the ideas of Plato when he wrote [7]:

> Our future lawyers, clergy, and statesmen are expected at the University to learn a good deal about curves, and angles, and number and proportions; not because these subjects have the smallest relation to the needs of their lives, but because in the very act of learning them they are likely to acquire that habit of steadfast and accurate thinking, which is indispensable in all the pursuits of life.

The current place of mathematics in the curricula around the world certainly embraces this idea. Differences in students' performance on mathematics exams is used to make claims about comparisons of entire nations' education systems, and individual exam scores are critically important factors in determining college acceptance.

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[3] B. Jowett, Ed., (1975) *The Dialogues of Plato.* Oxford Univ. Press, Oxford, p. 785.
[4] Suzanne, B. (2011). Frequently asked questions about plato: Let no one ignorant of geometry enter. Retrieved from http://plato-dialogues..org/faq/faq009.htm
[5] Richards, J. L. (2001). Connecting mathematics with reason. Mathematics and Democracy, 31.
[6] Dudley, U. (1999). What Is Mathematics For? *Notices of the AMS*, *57*(5), 608–613.
[7] *ibid.*

Building General Skills, "in General"

The discussion about the effects of learning mathematics on general higher-order cognitive skills can be placed within the larger question of the development of general cognitive skills. Only after establishing that it is in fact possible to teach higher-order thinking skills is it reasonable to ask if mathematics is the optimal, or even reasonable, domain for this process.

In a review paper on this topic, Perkins and Salomon[8] outline the development of the idea of teaching general thinking skills as an initial "golden age," a sharp transition to embracing the opposite extreme: the idea that all skills are completely contextual, and the slow climb toward a more nuanced understanding of the necessary conditions for the development of higher-order thinking skills. This development of the idea fit into the larger context of psychology research at the time, from initial excitement and intuitive claims, to behaviorism, which restricted psychology to studying only observable behavior rather than theorizing about mental processes, through the cognitive revolution.

At first, the idea of general heuristics was widely accepted, as it fit with the general notions of development of cognitive skills dating back to Plato. It is in the twentieth century, as scientific studies began empirically testing previously accepted beliefs, that this ancient idea first came under scrutiny. In these early studies, the fundamental idea of transfer between disciplines was criticized by Edward Thorndike[9] who devoted his career to empirically disproving this idea, testing people on a wide variety of tasks and finding very little transfer of learning between them. He claimed the extreme opposite to the ancient Platonic idea of abstract formal disciplines: all skills are bound by the contexts in which they were learned, and any transfer that successfully take place is a result of identical elements of the two tasks. This idea fit with and contributed to the rise of behaviorism at the time.

In parallel, a similar picture was forming in the research on the differences in reasoning of experts and novices. The main factors in experts' reasoning are "(a) a large database of domain-specific patterns, (b) rapid recognition of situations when these patterns apply, and (c) reasoning that moves directly from this recognition toward a solution by working with the patterns"[10]. It is not the case that in the transition from novice to expert one trains some abstract mental faculties, but rather she builds up a rich database of experience and learning patterns in each domain separately. The Artificial Intelligence (AI) community came to a similar conclusion: without a rich database, the computer could employ only general, "weak methods," which, as a rule, yield weak results. Research turned away from programs like the General Problem Solver and began developing highly context specific "expert systems."[11]

The argument for skills being entirely specific to context was powerful. Slowly, however, with more careful experiments, the prevailing opinion began moving back toward a middle ground.

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[8] Perkins, D., & Salomon, G. (1989). Are Cognitive Skills Context Bound? *Educational Researcher*, 16–25.
[9] Thorndike, E. (1906). *Principles of Teaching*. Seiler, New York. p.246
[10] Glaser, 1984 and Rabinowitz and Glaser, 1985 as cited in
Perkins, D., & Salomon, G. (1989). Are Cognitive Skills Context Bound? *Educational Researcher*, 16–25.
[11] Perkins, D., & Salomon, G. (1989). Are Cognitive Skills Context Bound? *Educational Researcher*, 16–25.

John Clement[12] showed that experts employ general heuristics in combination with their rich knowledge base in order to solve unfamiliar problems. Alan Schoenfeld[13] successfully taught college students heuristics that improved their mathematical problem solving skills by embedding them into the mathematics and pairing them with metacognitive practices. Graduate students in psychology and medicine (but not law or chemistry) are able to transfer their reasoning about methodology.[14] Transfer was even demonstrated among three- and four- year olds, when learners were provided with enough support (e.g. familiar domain, experimenters placed explicit emphasis on the underlying similarity of the two tasks, learners were encouraged to make rules, and placed in a social context).[15]

The result of this debate is a more nuanced understanding of the appropriate conditions for the development of transferable higher-order thinking skills. Research findings support the claim that general thinking skills can be taught and learned, but rely heavily on context specific domain knowledge and are most effective when presented as embedded within a domain rather than as independent principles. In line with the ideas of Jean Piaget and constructivism, training is more successful if students are prompted to create and experiment with the ideas thus building up their expertise, rather than being given the principles and asked to apply them.

Building Mathematics Skills in Particular

In Theory

The research described above suggests that higher-order thinking skills are best taught embedded in a specific context rather than in isolation. The question remains, however, of why the field of mathematics has been credited with being the best domain for this goal.

One idea is the “purity” and rigor of mathematics as a discipline. Because math is so abstract, the mathematician cannot rely on her usual assumptions that are embedded in domain knowledge; instead, she is forced define her terms, to abandon those intuitions which do not fit into the mathematics, and to prove her ideas using only the tools of logic. Both throughout the history of science and in its current application, mathematics is used to formalize reasoning that is otherwise based on intuition and thus may contain biases and other flaws; math is the ultimate rigorous study and successfully applying it to a field or real world problem grants that domain rigor as well. This echoes the ideas of Plato about teaching math for its own sake, rather than to serve more concrete and practical domains. The implicit connection to education is that theoretically, if we want students to learn to think rigorously, we should teach them the subject that exemplifies rigor by stripping away everything else.

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[12] Clement, J. (1982). Analogical reasoning patterns in expert problem solving. Proceedings of the Fourth Annual Conference of the Cognitive Science Society. Ann Arbor, MI: University of Michigan.
[13] Schoenfeld, A. H. (1982). Measures of problem-solving performance and of problem-solving instruction. Journal for Research in Mathematics Education, 13(1), 31-49.
[14] Lehman, D. R., Lempert, R. O., & Nisbett, R. E. (1988). The effects of graduate training on reasoning: Formal discipline and thinking about everyday-life problems. American Psychologist, 43, 431-442.
[15] Brown, A. L., & Kane, M. J. (1988) Cognitive flexibility in young children: the case for transfer. Symposium paper presented at the Annual Meeting of the American Educational Research Association, New Orleans, LA.

However, juxtaposing this characterization of mathematics with the description of the necessary conditions for teaching transferable higher-order reasoning skills discussed above highlights a disturbing misalignment. While successful training of general cognitive skills requires a concrete and familiar knowledge base, mathematics is the ultimate abstract and unfamiliar one. The same reasons that make mathematics a powerful and rigorous tool when it is mastered, make it potentially a very difficult domain for the teaching of higher-order thinking skills.

In the Brain

With the advent of new technologies researchers have been able to move past sidestepping questions about what goes on in the mind (as in behaviorism) or theorizing about it (as in cognitive psychology), and measure empirically some physiological aspects of the brain. Most of the research on mathematics in the brain has focused on infants' innate number sense, children learning to count, the effect of formal education on mathematical thinking, representation of numerical processing in the brain, disabilities, and anxiety.

There have not been studies that tease out the broader effects of learning mathematics on the more general mental functions, perhaps because of the methodological challenges of isolating this causative link. Interestingly, the converse has been studied somewhat more extensively; it turns out that higher-level domain general cognitive processes such as executive function, inhibitory control, attention shifting, and working memory are all critical for mathematical cognition and support development of more efficient strategies during the early stages of arithmetic skill acquisition. For example, preschoolers' visual-spatial short term memory span and working memory predicted their mathematical achievement in the third grade, and executive functioning of the preschoolers predicted their general academic achievement in third grade.[16] This suggests that domain general skills support the development of more finely tuned skills like those necessary for mathematical achievement.

In fact, the developmental trajectory for children learning mathematics appears to start with more general processing that becomes more specially tuned with practice. In particular, there is a decrease of activation of the frontal cortex, the hippocampus, and the basal ganglia, and an increase in the left inferior parietal cortex, anterior intraparietal sulcus and left lateral temporo-occipital cortex.[17] This suggests both automaticity (and thus less reliance on attention and working memory) and consolidation of knowledge into long term memory.

It may be possible to learn about the effect of mathematics on the brain by comparing the brains of mathematicians with non-mathematicians, because the difference would presumably reflect the effects of studying mathematics. Research has shown that mathematicians had higher

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[16] Bull, R., Espy, K.A., and Wiebe, S.A. (2008). Short-Term Memory, Working Memory, and Executive Functioning in Preschoolers: Longitudinal Predictors of Mathematical Achievement at Age 7 Years. Developmental Neuropsychology 33(3), 205-228.

[17] Rivera, S.M., A.L. Reiss, M.A. Eckert, and V. Menon. 2005. Developmental changes in mental arithmetic: Evidence for increased specialization in the left inferior parietal cortex. *Cerebral Cortex* 15: 1779–90.

grey matter density in the bilateral inferior parietal lobe and left inferior frontal gyrus.[18] These findings suggest that through practice, the mathematicians trained the brain regions associated with numerical cognition; there is no evidence here that in doing so they also trained their higher-order cognitive skills. Together with the findings of increased specialization and decreased reliance on general higher-order skills with mathematical development, this suggests that mathematics training is not especially powerful in the training of higher-order cognitive skills.

Of course, for each mathematics problem there are several different successful strategies, and each one works differently in the brain. Perhaps some strategies *do* in fact train higher-order thinking skills while others do not. The choice of strategy depends both on the problem and the individual's cognitive capacities. Even children in kindergarten possess a variety of strategies to solve addition problems,[19] but the general trend of development moves from effortful counting, to a variety of other strategies -- e.g. 'min strategy' (counting from the larger addend)[20] or 'tie-strategy' (2+2, 3+3, 4+4) -- to direct retrieval.[21] In other words, with more formal education, children learn to optimize their strategy choice.

The developmental trajectory of strategy optimization as one from effortful counting, to a variety of strategies, ultimately to direct retrieval, reveals an interesting point about the kinds of problems being presented and what their successful solving involves for the brain. Problems that are best solved through direct retrieval from long term memory may not be the kinds of problems that develop higher order cognitive skills. These kinds of problems best fit under the perspective of behaviorism, ignoring and thus not exercising higher-level processes in the mind. In fact, Thorndike acknowledged this and advocated for drill exercises to teach mathematics, believing that it was simply a matter of tuning the correct question and answer pairs in the brains of the students through repetition.[22]

On the cognitive side of the debate, however, there are those who advocate for "meaningful learning"[23] in line with the constructivist Piagetian line of thought. Research has shown that learning by drill can make retrieval of facts faster, while meaningful learning increases transfer of learning to new problems.[24] David Sousa, in his book *How the Brain Learns Mathematics*, emphasizes that drills lead to mindless application of procedures in contexts, often incorrectly, while meaningful learning allows students to understand how the system works, not just to get the right answer quickly.[25]

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[18] Aydin, K., Ucar, A., Oquz, K.K., Okur, O.O., Agayev, A., Unal, Z., Yilmaz, S., and Ozturk, C. (2007). Increased gray matter density in the parietal cortex of mathematicians: a voxel---based morphometry study. American Journal of Neuroradiology 28, 1859---1864.

[19] Groen, G., & Resnick, L. B. (1977). Can preschool children invent addition algorithms? *Journal of Educational Psychology*, *69*(6), 645–652.

[20] *ibid.*

[21] Barrouillet, P., and Fayol, M. (1998). From algorithmic computing to direct retrieval: evidence from number and alphabetic arithmetic in children and adults. Memory & cognition 26, 355---368.

[22] Thorndike, E.L. (1922). The psychology of arithmetic. New York: The Macmillan Co.

[23] Brownell, W.A. (1935). Psychological considerations in the learning and the teaching of arithmetic. The teaching of arithmetic. The tenth yearbook of the National Council of Teachers of Mathematics. New York: Teachers College, Columbia University.

[24] Resnick, L.B., & Ford, W.W. (1981). The psychology of mathematics for instruction. Hillsdale, NJ: Lawrence Erlbaum Associates.

[25] Sousa, D. A. (2008). *How the Brain Learns Mathematics*. Thousand Oaks, CA: Corwin Press.

The seminal work of Piaget[26] helps to bridge the theoretical value of mathematics with the development of children's mathematical abilities. He showed that children begin by believing that the amount/number of a substance/object changes when the perceptual features seem to change (such as pouring liquid into a tall but skinny glass or spreading the objects on a table), and only around the age of seven do they learn mathematical ideas such as conservation of number. In doing so, they successfully abstract from their perceptions (e.g. 'it looks like there is more') and learn to think more rigorously about the comparison problems they are presented with. Piaget emphasizes the constructivist model of learning: that children must build up their own knowledge of the world through active participation in it; telling them what to memorize will not result in a thorough and useful understanding of the world.

One concrete example of abstracting away from the perceptual features of one's experience is the mental models of numbers children use in estimation tasks. Children have several ways (linear and logarithmic) of representing number lines in their minds that they can call upon to perform estimation tasks. Context determines which will be picked for a particular problem, and with experience they learn to rely on the appropriate representations for the corresponding tasks.[27] This suggests that by with mathematics training, children are able to abstract away from their original associations based on perceptual features and focus on the optimal strategy for estimation. To test the role of education, a microgenetic study of 93 second graders making this transition and found that feedback was extremely useful in their progress, with change often occurring at the level of the entire representation, and the larger the original discrepancy between the predictions of the two mental models, the larger the change.[28] In fact, Stanislas Dehaene has found that exact arithmetic calculations employ language-specific representations in the left inferior prefrontal cortex and in this form they are difficult to transfer to other domains, whereas approximate calculations activate bilateral parietal visuo-spatial networks which are language independent.[29]

Compared to the previously described simple tasks of addition in which children also learned to optimize their strategy choice, finally relying on direct retrieval from memory as the optimal choice, this strategy optimization for estimation happens at a more basic level; although children are not consciously thinking about testing their different mental models against their experience with the world, their brain is doing a similar process. Since they cannot eventually rely on direct retrieval, they continue to optimize their strategy selection and automaticity.

Perhaps the source of the confusion about whether mathematics is a suitable topic for teaching higher-order thinking skills lies in the differing ideas about what the mathematics educational experience is. Surely some tasks that fall under the overarching term "school mathematics" do not train higher-order thinking, but it could still be true that sophisticated ideas about proof, taught well, do in fact have this effect. In a seminal work, Fawcett compared teaching

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[26] Piaget, J. (1952). The child's conception of number. London: Routledge & Kegan Paul.
[27] Siegler, R. S., & Opfer, J. E. (2010). Research Article The Development of Numerical Estimation: Evidence for Multiple Representations of Numerical Quantity, *14*(3), 237–243.
[28] Opfer, J. E., & Siegler, R. S. (2007). Representational change and children's numerical estimation. *Cognitive psychology*, *55*(3), 169–95.
[29] Dehaene, S. (1999). Sources of Mathematical Thinking: Behavioral and Brain-Imaging Evidence. *Science*, *284*(5416), 970–974.

geometric proofs in a constructivist, student-driven manner with the traditional method of teaching.[30] After the two years of the study, the experimental group of students outperformed the control group in measures of geometry knowledge, retention, and transfer of reasoning to real life situations. Unfortunately, there were several design features of the experiment that were problematic so it is difficult to draw far-reaching conclusions. Due to the nuanced nature of teaching, rigorous experimentation on this topic is extremely difficult to execute. Studies like this one, however, point to teaching as a potentially important factor in determining the degree to which, if at all, students develop higher order cognitive skills while learning mathematics. Future research should address these questions methodically, rigorously, and making full use of new technologies where appropriate.

There is some evidence that indicate that studying how the brain performs various mathematical tasks could lead to a greater understanding of the processes involved, and the circumstances under which studying mathematics does in fact enhance higher-order cognitive skills. New brain stimulation methodologies have shown some evidence for a trade-off between automaticity and acquisition of new artificial numerical symbols. Stimulating the parietal cortex of individuals undergoing training increased acquisition but decreased automaticity, while stimulating the dorsolateral prefrontal cortex (DLPFC) resulted in the reverse pattern. While the authors stress that it these findings may not be generalizable to other cognitive activities or methodologies, they do suggest that cognitive benefits may come at unexpected costs, and thus begin the conversation about trade off relationships between different cognitive functions at different levels that are involved in learning. [31] Stimulating the DLPC with random noise enhanced the speed of both memory-based and calculation-based arithmetic tasks. These improvements were apparent in the more efficient hemodynamic response of the left DLPFC as measured by near-infrared spectroscopy (NIRS), and were still apparent six months after the initial treatment.[32] Future studies should build on these findings and carefully tease out the roles of the various brain areas and the effects of different kinds of mathematical thinking.

Taken together, these studies do explore many different facets of mathematics in the brain, but they do not provide evidence that mathematics training is uniquely suited to build higher order functions. There is clearly a relationship between mathematics achievement, the DLPFC, the IPS, and general cognitive skills such as executive function and working memory, but ***from the current findings it is most likely that higher order functions support the building of mathematics skills, not the reverse.***

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[30] Fawcett, H. P. (1938). The nature of proof: A description and evaluation of certain procedures used in a senior high school todevelop an understanding of the nature of proof. New York: Teachers College, Columbia University.

[31] Iuculano, T., & Cohen Kadosh, R. (2013). The mental cost of cognitive enhancement. *The Journal of neuroscience : the official journal of the Society for Neuroscience*, *33*(10), 4482–6.

[32] Snowball, A., Tachtsidis, I., Popescu, T., Thompson, J., Delazer, M., Zamarian, L., Zhu, T., et al. (2013). Long-term enhancement of brain function and cognition using cognitive training and brain stimulation. *Current biology : CB*, *23*(11), 987–92.

Conclusion

We have shown here that there is not sufficient evidence to conclude that mathematics enhances higher order cognitive functions. The CCR calls for a much stronger cognitive psychology and neuroscience research base to be developed on the effects of studying mathematics. Future research should aim to disentangle the brain processes involved in various mathematical tasks, teaching styles and epistemologies, and determine which of them, if any, could train higher-order thinking.