Street-fighting mathematics for everyone

Sanjoy Mahajan

Olin College of Engineering

streetfightingmath.com
sanjoy@olin.edu

21st Century Mathematics, Stockholm, 24 April 2013

Students can solve problems they don't understand

Write a story problem for

$6 \times 3 =$ ____________

Students can solve problems they don't understand

Write a story problem for

$6 \times 3 =$ ____________

Most common answer type in 4th and 5th grades: *There were six ducks swimming in a pond. Then a while later three more ducks come so how many are there? Six times three is eighteen. That's the answer.*

Grade 4	37%
Grade 5	44%

Students can solve problems they don’t understand

There are 26 sheep and 10 goats on a ship. How old is the captain?

Students can solve problems they don't understand

There are 26 sheep and 10 goats on a ship. How old is the captain?

36

Rote learning is the result of most education. Instead, teach street-fighting reasoning

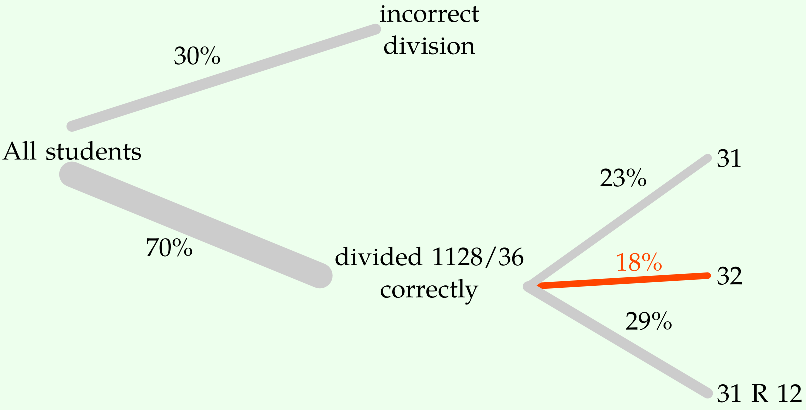
1. Rote learning and its consequence
2. Street-fighting tools
   a. lumping
   b. comparing

Students divide without understanding

An army bus holds 36 soldiers. If 1128 soldiers are being bused to their training site, how many buses are needed?

Students divide without understanding

An army bus holds 36 soldiers. If 1128 soldiers are being bused to their training site, how many buses are needed?



Using a calculator harmed students' performance

An army bus holds 36 soldiers. If 1128 soldiers are being bused to their training site, how many buses are needed?

	calculator	*paper/pencil*
right	18 (7.2%)	59 (23.6%)
wrong	232	191

Using a calculator harmed students' performance

An army bus holds 36 soldiers. If 1128 soldiers are being bused to their training site, how many buses are needed?

	calculator	*paper/pencil*
right	18 (7.2%)	59 (23.6%)
wrong	232	191

$P(\text{calculator helped or did no harm} \mid \text{data}) \approx 10^{-7}$.

Students need to turn on their minds, not their calculator

Estimate 3.04×5.3

1.6
16
160
1600
No answer

Students need to turn on their minds, not their calculator

Estimate 3.04×5.3

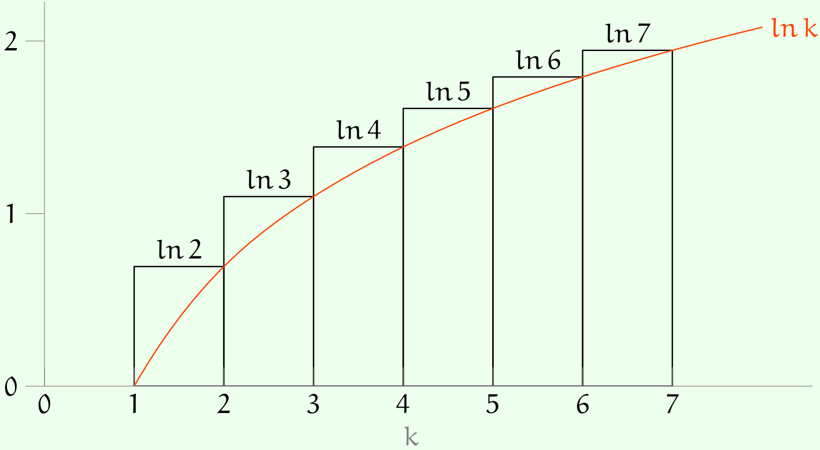
	Age 13
1.6	28%
16	21
160	18
1600	23
No answer	9

Students need to turn on their minds, not their calculator

Estimate 3.04×5.3

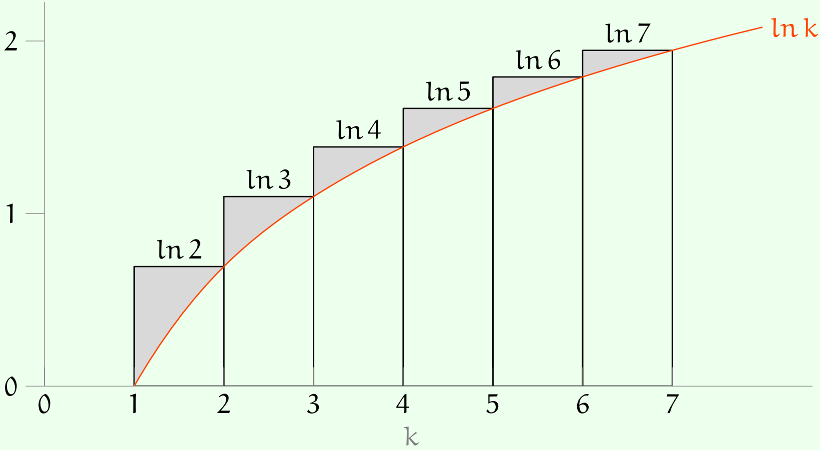
	Age 13	*Age 17*
1.6	28%	21%
16	21	37
160	18	17
1600	23	11
No answer	9	12

Rote learning happens at all educational levels



Is $\ln 7!$ greater than or less than $\int_1^7 \ln k \, dk$?

Rote learning happens at all educational levels



Is $\ln 7!$ greater than or less than $\int_1^7 \ln k \, dk$?

Rote learning happens at all educational levels

Students reasoned using only numerical calculation:

$$\int_1^7 \ln k \, dk = k \ln k - k \Big|_1^7 \approx 7.62.$$

$$\ln 7! = \sum_1^7 \ln k \approx 8.52.$$

$$\underbrace{8.52}_{\sum} \quad > \quad \underbrace{7.62}_{\int}$$

Rote learning combines the worst of human and computer thinking

	human chess	*computer chess*
calculation	1 position/second	10^8 positions/second
judgment	fantastic	minimal

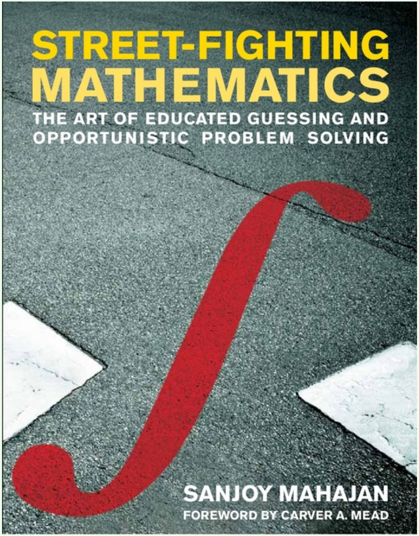
Rote learning combines the worst of human and computer thinking

	human chess	*computer chess*
calculation	1 position/second	10^8 positions/second
judgment	fantastic	minimal

Rote learning is the result of most education. Instead, teach street-fighting reasoning

1. Rote learning and its consequence
2. Street-fighting tools
   a. lumping
   b. comparing

Street fighting is the pragmatic opposite of rigor



MIT Press, 2010

Freely, and legally, available from MIT Press—with freedom to modify or redistribute

Street fighting is the pragmatic opposite of rigor

Rigor

Street fighting is the pragmatic opposite of rigor

Rigor mortis

Street-fighting tool 1: Simplify using lumping

Every number is of the form:

$$\begin{pmatrix} \text{one} \\ \textit{or} \\ \text{few} \end{pmatrix} \times 10^n,$$

where

$$\text{few}^2 = 10.$$

Street-fighting tool 1: Simplify using lumping

How many seconds in a year?

$$\frac{365\,\text{days}}{\text{year}} \times \frac{24\,\text{hours}}{\text{day}} \times \frac{3600\,\text{seconds}}{\text{hour}}$$

Street-fighting tool 1: Simplify using lumping

How many seconds in a year?

$$\frac{\text{few} \times 10^2\ \text{days}}{\text{year}} \times \frac{24\ \text{hours}}{\text{day}} \times \frac{3600\ \text{seconds}}{\text{hour}}$$

Street-fighting tool 1: Simplify using lumping

How many seconds in a year?

$$\frac{\text{few} \times 10^2\,\text{days}}{\text{year}} \times \frac{\text{few} \times 10^1\,\text{hours}}{\text{day}} \times \frac{3600\,\text{seconds}}{\text{hour}}$$

Street-fighting tool 1: Simplify using lumping

How many seconds in a year?

$$\frac{\text{few} \times 10^2\,\text{days}}{\text{year}} \times \frac{\text{few} \times 10^1\,\text{hours}}{\text{day}} \times \frac{\text{few} \times 10^3\,\text{seconds}}{\text{hour}}$$

Street-fighting tool 1: Simplify using lumping

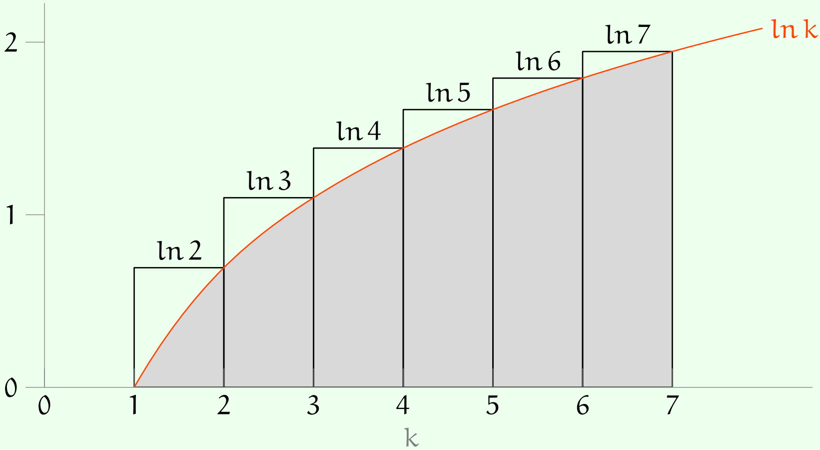
How many seconds in a year?

$$\frac{\text{few} \times 10^2\,\text{days}}{\text{year}} \times \frac{\text{few} \times 10^1\,\text{hours}}{\text{day}} \times \frac{\text{few} \times 10^3\,\text{seconds}}{\text{hour}}$$

$$\sim \text{few} \times 10^7\,\frac{\text{seconds}}{\text{year}}$$

Lumping also works on graphs

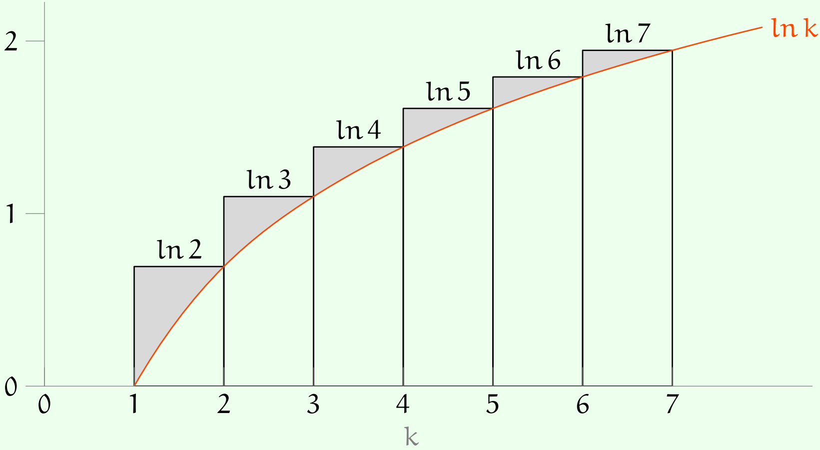
Pictures explain most of Stirling's formula for $n!$



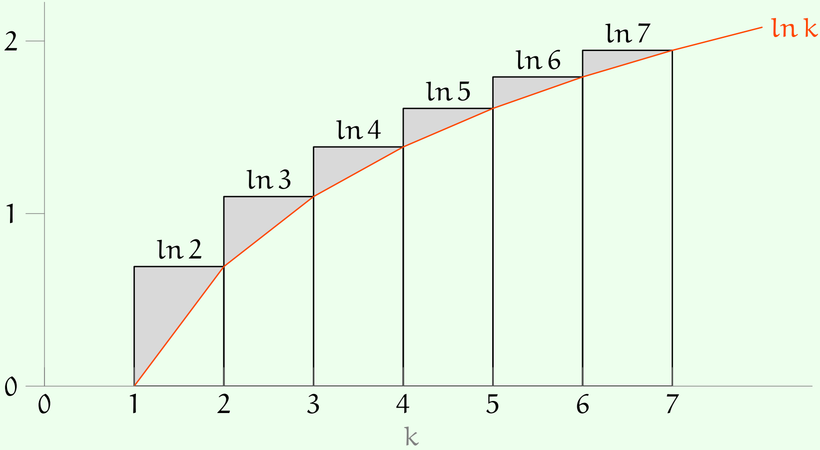
$$\ln n! \approx \int_1^n \ln k \, dk = n \ln n - n + 1;$$

$$n! \approx e \times n^n / e^n.$$

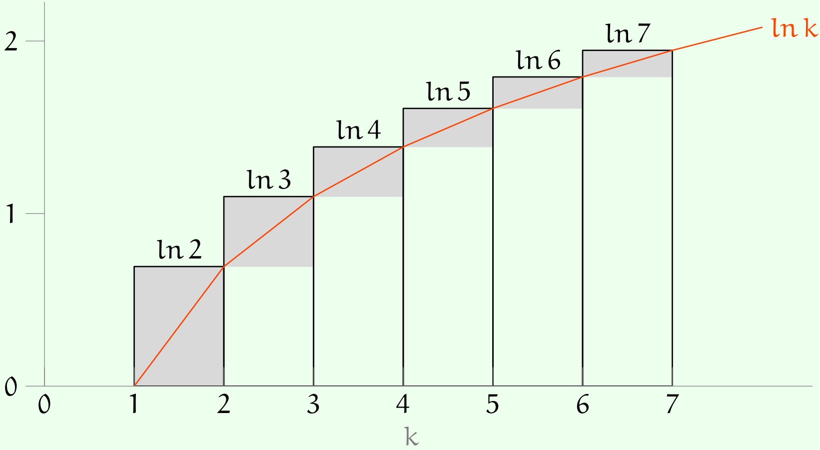
The protrusions are the underestimate



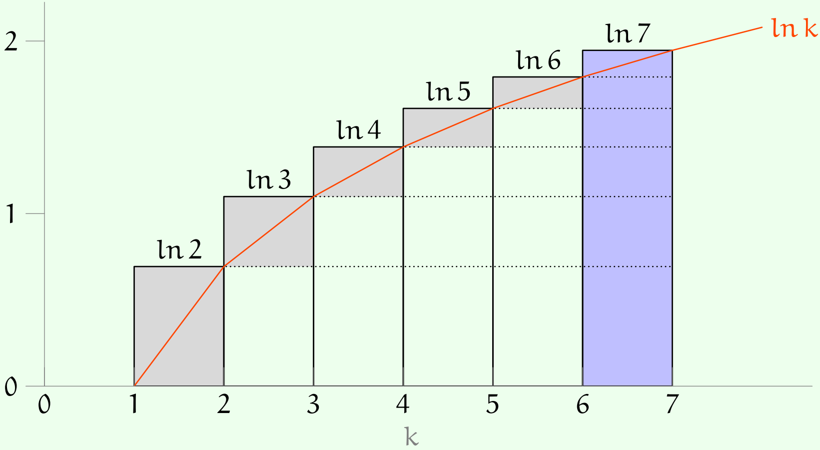
Each protrusion is almost a triangle



Doubling each triangle makes them easier to add



The doubled triangles stack nicely

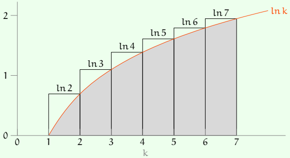


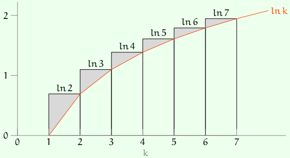
Sum of doubled triangles = $\ln n$

The integral along with the triangles explain most pieces of Stirling's formula for $n!$

$$\ln n! = \sum_{1}^{n} \ln k$$

$$\approx \quad \underbrace{n \ln n - n + 1} \quad + \quad \underbrace{\frac{1}{2} \ln n}$$





$$n! \approx \underbrace{e}_{\text{should be } \sqrt{2\pi}} \times \; n^n / e^n \times \sqrt{n}$$

Rote learning is the result of most education. Instead, teach street-fighting reasoning

1. Rote learning and its consequence
2. Street-fighting tools
    a. lumping
    b. comparing

Hard problems demand more street-fighting methods

What is the fuel efficiency of a 747?

The rote method is hopelessly difficult

Equations of fluid mechanics

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = -\frac{1}{\rho}\boldsymbol{\nabla} p + \nu\boldsymbol{\nabla}^2\mathbf{v}$$

$$\boldsymbol{\nabla}\cdot\mathbf{v} = 0$$

where

$\rho =$ air density

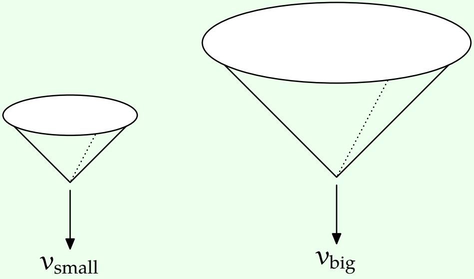
$p =$ pressure

$\mathbf{v} =$ velocity

$\nu =$ (kinematic) viscosity

$t =$ time

Pull out street-fighting tool 2: Proportional reasoning



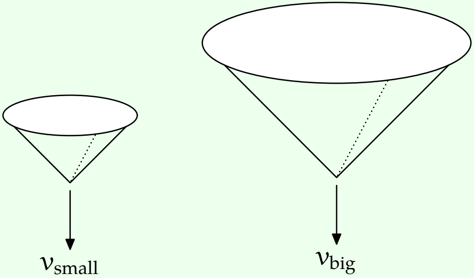
What is the approximate ratio of the fall speeds $v_{\text{big}}/v_{\text{small}}$?

a. 2 : 1

b. 1 : 1

c. 1 : 2

Pull out street-fighting tool 2: Proportional reasoning



What is the approximate ratio of the fall speeds $v_{\text{big}}/v_{\text{small}}$?

a. 2 : 1

b. 1 : 1 Drag force is proportional to area!

c. 1 : 2

We need a short interlude with a symmetry principle

$$\underbrace{\text{drag force}}_{\frac{\text{kilograms}\times\text{meters}}{\text{second}^2}} \sim \underbrace{\text{area}}_{\text{meters}^2} \times \underbrace{\text{density? speed? viscosity?}}_{\frac{\text{kilograms}}{\text{meter}\times\text{second}^2}}$$

We need a short interlude with a symmetry principle

$$\underbrace{\text{drag force}}_{\frac{\text{kilograms} \times \text{meters}}{\text{second}^2}} \sim \underbrace{\text{area}}_{\text{meters}^2} \times \underbrace{\text{density}}_{\frac{\text{kilograms}}{\text{meter}^3}} \times \underbrace{\text{speed? viscosity?}}_{\frac{\text{meters}^2}{\text{second}^2}}$$

We need a short interlude with a symmetry principle

$$\underbrace{\text{drag force}}_{\dfrac{\text{kilograms} \times \text{meters}}{\text{second}^2}} \sim \underbrace{\text{area}}_{\text{meters}^2} \times \underbrace{\text{density}}_{\dfrac{\text{kilograms}}{\text{meter}^3}} \times \underbrace{\text{speed}^2}_{\dfrac{\text{meters}^2}{\text{second}^2}}$$

Return to proportional reasoning

Fuel consumption is proportional to the drag force, and

drag force $\sim$ area $\times$ density $\times$ speed^2.

The ratio of plane-to-car fuel consumptions is therefore

$$\frac{\text{plane consumption}}{\text{car consumption}} \sim \underbrace{\frac{\text{area}_{\text{plane}}}{\text{area}_{\text{car}}}}_{?} \times \underbrace{\frac{\text{density}_{\text{plane}}}{\text{density}_{\text{car}}}}_{?} \times \underbrace{\frac{\text{speed}^2_{\text{plane}}}{\text{speed}^2_{\text{car}}}}_{?}$$

Return to proportional reasoning

Fuel consumption is proportional to the drag force, and

drag force $\sim$ area $\times$ density $\times$ speed^2.

The ratio of plane-to-car fuel consumptions is therefore

$$\frac{\text{plane consumption}}{\text{car consumption}} \sim \underbrace{\frac{\text{area}_{\text{plane}}}{\text{area}_{\text{car}}}}_{10} \times \underbrace{\frac{\text{density}_{\text{plane}}}{\text{density}_{\text{car}}}}_{?} \times \underbrace{\frac{\text{speed}^2_{\text{plane}}}{\text{speed}^2_{\text{car}}}}_{?}$$

Return to proportional reasoning

Fuel consumption is proportional to the drag force, and

drag force $\sim$ area $\times$ density $\times$ speed^2.

The ratio of plane-to-car fuel consumptions is therefore

$$\frac{\text{plane consumption}}{\text{car consumption}} \sim \underbrace{\frac{\text{area}_{\text{plane}}}{\text{area}_{\text{car}}}}_{10} \times \underbrace{\frac{\text{density}_{\text{plane}}}{\text{density}_{\text{car}}}}_{1/3} \times \underbrace{\frac{\text{speed}^2_{\text{plane}}}{\text{speed}^2_{\text{car}}}}_{?}$$

Return to proportional reasoning

Fuel consumption is proportional to the drag force, and

drag force $\sim$ area $\times$ density $\times$ speed^2.

The ratio of plane-to-car fuel consumptions is therefore

$$\frac{\text{plane consumption}}{\text{car consumption}} \sim \underbrace{\frac{\text{area}_{\text{plane}}}{\text{area}_{\text{car}}}}_{10} \times \underbrace{\frac{\text{density}_{\text{plane}}}{\text{density}_{\text{car}}}}_{1/3} \times \underbrace{\frac{\text{speed}^2_{\text{plane}}}{\text{speed}^2_{\text{car}}}}_{100}$$

Return to proportional reasoning

Fuel consumption is proportional to the drag force, and

drag force $\sim$ area $\times$ density $\times$ speed^2.

The ratio of plane-to-car fuel consumptions is therefore

$$\frac{\text{plane consumption}}{\text{car consumption}} \sim \underbrace{\frac{\text{area}_{\text{plane}}}{\text{area}_{\text{car}}}}_{10} \times \underbrace{\frac{\text{density}_{\text{plane}}}{\text{density}_{\text{car}}}}_{1/3} \times \underbrace{\frac{\text{speed}^2_{\text{plane}}}{\text{speed}^2_{\text{car}}}}_{100} \sim 300$$

Return to proportional reasoning

Fuel consumption is proportional to the drag force, and

drag force $\sim$ area $\times$ density $\times$ speed^2.

The ratio of plane-to-car fuel consumptions is therefore

$$\frac{\text{plane consumption}}{\text{car consumption}} \sim \underbrace{\frac{\text{area}_{\text{plane}}}{\text{area}_{\text{car}}}}_{10} \times \underbrace{\frac{\text{density}_{\text{plane}}}{\text{density}_{\text{car}}}}_{1/3} \times \underbrace{\frac{\text{speed}^2_{\text{plane}}}{\text{speed}^2_{\text{car}}}}_{100} \sim 300.$$

But 300 passengers on a plane flight; only 1 passenger in a car.

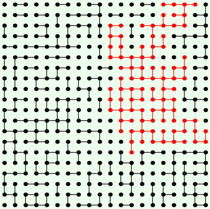
Planes and cars are equally fuel efficient!

The connection between falling cones and flying planes helps us estimate the cost of a plane ticket

A New York–Stockholm roundtrip is roughly $12,000\,\mathrm{km}$.

$$12,000\,\mathrm{km} \times \frac{8\,\mathrm{litres}}{100\,\mathrm{km}} \times \frac{0.5\,\mathrm{euros}}{1\,\mathrm{litre}} \sim 500\,\mathrm{euros}.$$

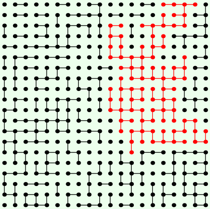
Connections are more important than facts alone



big cluster = 22%

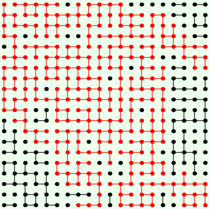
$p_{\text{bond}} = 0.40$

Connections are more important than facts alone



big cluster = 22%

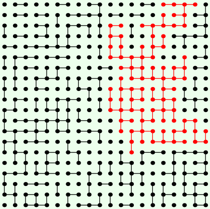
$p_{\text{bond}} = 0.40$



big cluster = 68%

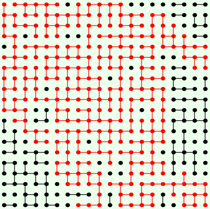
$p_{\text{bond}} = 0.50$

Connections are more important than facts alone



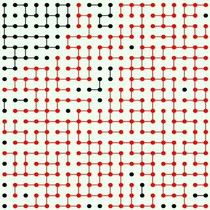
big cluster = 22%

$p_{\text{bond}} = 0.40$



big cluster = 68%

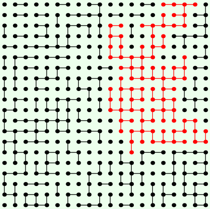
$p_{\text{bond}} = 0.50$



big cluster = 80%

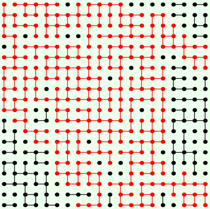
$p_{\text{bond}} = 0.55$

Connections are more important than facts alone



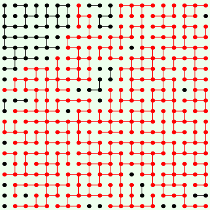
big cluster = 22%

$p_{bond} = 0.40$



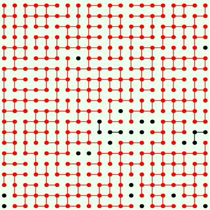
big cluster = 68%

$p_{bond} = 0.50$



big cluster = 80%

$p_{bond} = 0.55$



big cluster = 93%

$p_{bond} = 0.60$

Rote learning is the result of most education. Instead, teach street-fighting reasoning

The goal [of teaching] should be, not to implant in the students' mind every fact that the teacher knows now;

but rather to implant a *way of thinking* that enables the student, in the future, to learn in one year what the teacher learned in two years.

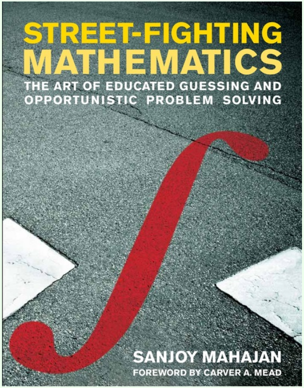
Only in that way can we continue to advance from one generation to the next.

—*Edwin T. Jaynes (1922–1998)*

Street-fighting mathematics for everyone

Sanjoy Mahajan
Olin College of Engineering

streetfightingmath.com
sanjoy@olin.edu



Produced with free software:
Maxima, PDFTEX, ConTEXt, and MetaPost

21st Century Mathematics, Stockholm, 24 April 2013