CENTER FOR CURRICULUM REDESIGN

Mathematics for the 21st Century

Paper #3

WHAT should students learn?

Concepts and processes

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What is the goal of this project?

This project aims at defining what mathematics education should look like in the 21st century. To do so, we will attempt at establishing a synthetic **ontology of mathematics**. CCR considers that it might be possible to show what Mathematics *could* be teaching in terms of concepts - concepts that people ought to remember even after they have forgotten how to precisely manipulate *processes or tools.* It is our thesis that concepts and processes should be stressed in mathematics education instead of rote procedural knowledge, as described in paper #2 of this series. Therefore, the first step, which is the focus of this paper, is to establish a short list of such *concepts* and *processes*.

The Center for Curriculum Redesign would like to thank Pr. Zalman Usiskin for his precious help in identifying the concepts and processes shortlisted in this paper.

Challenges faced – Dowling & Dudley

In identifying useful mathematical concepts, the questions of what we mean by mathematics and what is the purpose of a mathematics education comes front and center. Ernest[1] identified in the late 1990s five different groups contesting the nature and aims of mathematics curriculum:

1. *Industrial Trainer aims* - 'back-to-basics': numeracy and social training in obedience (authoritarian),
2. *Technological Pragmatist aims* - useful mathematics to the appropriate level and knowledge and skill certification (industry-centered),
3. *Old Humanist aims* - transmission of the body of mathematical knowledge (mathematics-centered),
4. *Progressive Educator aims* - creativity, self- realization through mathematics (child-centered),
5. *Public Educator aims* - critical awareness and democratic citizenship via mathematics (social justice centered).

Several, if not all of those groups' philosophies rely on the assumption that mathematical concepts are observable in daily tasks and that a mastery of mathematics could help better accomplish those tasks. However, research has brought elements of reflection allowing us to question this assumption.

Dowling[2] noted in his *Sociology of Mathematics Education* that the ability to formulate a problem using mathematical concepts did not necessarily mean that mathematics was involved or that it could help better solve the problem. One might think indeed, that a

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[1] Ernest, P. (2002). *The philosophy of mathematics education*. Psychology Press.

[2] Dowling, P. (2002). *The sociology of mathematics education: Mathematical myths/pedagogic texts*. Routledge.

factory workers sorting objects based on their shape and size on a production line is doing nothing but algorithmic and geometry. Proving however that a good understanding of the related concepts can help the worker better do his job may turn out rather difficult.

Dudley[3] also tackled in an AMS notice the question of the use of mathematics outside of the academic and school world, arguing that people claiming to be using mathematics were often mistaken, or were not at least using mathematics to a degree justifying the possession of a strong mathematical background.

Scientific literature dealing with both purpose and usefulness of mathematics education do not deny the need for mathematicians in society. What is questioned nonetheless is rather the degree to which mathematical literacy is necessary on a global scale, and the fulfillment of its expectation.

Challenges faced by mathematics education seem both *cultural* and *sociological*:
On one hand, mathematics education must cultivate *mathematical awareness* for people to acknowledge when they are unknowingly doing or could knowingly use mathematics to solve a given problem or accomplish a given task. However, this cannot be done without taking into account socio-cultural factors: as there is no culturally neutral way to do mathematics, mathematical formalism will remain embedded into a specific culture and social environment, creating a potential formalism and reasoning gap across cultures and social classes.
On the other hand, mathematics education must not neglect the sociological *appropriateness* of its teaching: solutions, approaches and reasoning promoted by mathematics education must take into account alternatives to mathematics developed by people to avoid, not necessarily knowingly or intentionally, using mathematics. This problem of sociological *appropriateness* relates back to the issue mentioned before - the possibility to characterize a problem mathematically without mathematics being necessarily neither a pertinent solution nor a pertinent approach to adopt.
Failure to take into account those challenges will put mathematics education at risk of remaining a purely intellectual exercise disconnected from people's daily experiences. To be efficient, mathematics education must indeed not only develop specific mathematical skills and knowledge but also reduce the gap that may exist between schooling and non-academic environments by teaching how to connect daily experiences with mathematics when appropriate and hopefully help combat negative affect experiences (fear of maths).

Keeping those issues in mind, one could suggest an optimistic approach relying on the transmission of interdisciplinary concepts modeled along what people deal with, both practically and conceptually, on a daily-basis. Could it be that by reducing the constraints set by the abstract formalism of mathematics education, and instead focusing on the teaching

[3] Dudley, U. (2010). What Is Mathematics For?. *Notices of the AMS*, *57*(5), 608-613.

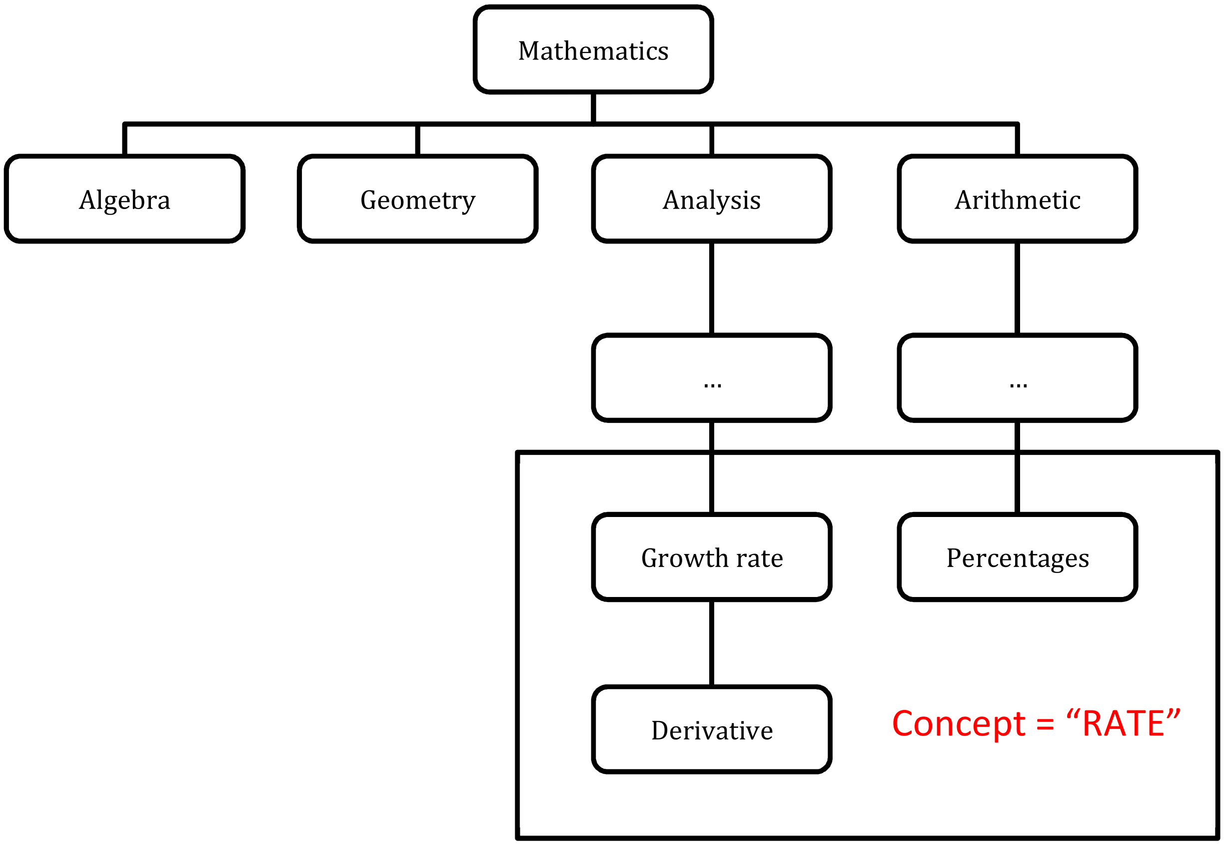
of those concepts and processes, leave a track accompanying people all along their intellectual life? And so, long after they have lost the mastery of mathematics acquired through their academic and schooling experiences.

What do we mean by concepts and processes?

- *Concepts* are defined as notions/abstract ideas.

Considering the ensemble of mathematical concepts represented in a tree, with at its top the concept “mathematics”, we try to describe the highest level possible in order to have the most synthetic list covering the more notions. We do not mention what could be called *branches* (such as “algebra”, “geometry”, etc.) and only list concepts, for clarity’s sake. Our goal is to produce a list as short as possible of self-explanatory concepts suggesting as many sub-concepts as possible.

Example:
In the following example, we illustrate a traditional representation of mathematical concepts in western curriculum (see references). Under mathematics are represented four categories (non-exhaustive): algebra, geometry, analysis and probabilities. All those categories will then introduce mathematical concepts and sub-concepts. Our approach gathers sub-concepts together across categories in order to reduce the number of concepts listed and adopt a trans-category approach. For instance, we find that the concepts of percentages in probabilities and of growth rate and derivative in analysis have in common to deal with *rates* (derivative being a growth rates variation). They will not appear in our list of concepts but will be suggested by the concept “rate”.



- *Processes* are to be understood as methodologies or know-how related to the manipulation of the listed concepts. These are designed not to remain field-specific and should therefore be applicable outside of mathematics.

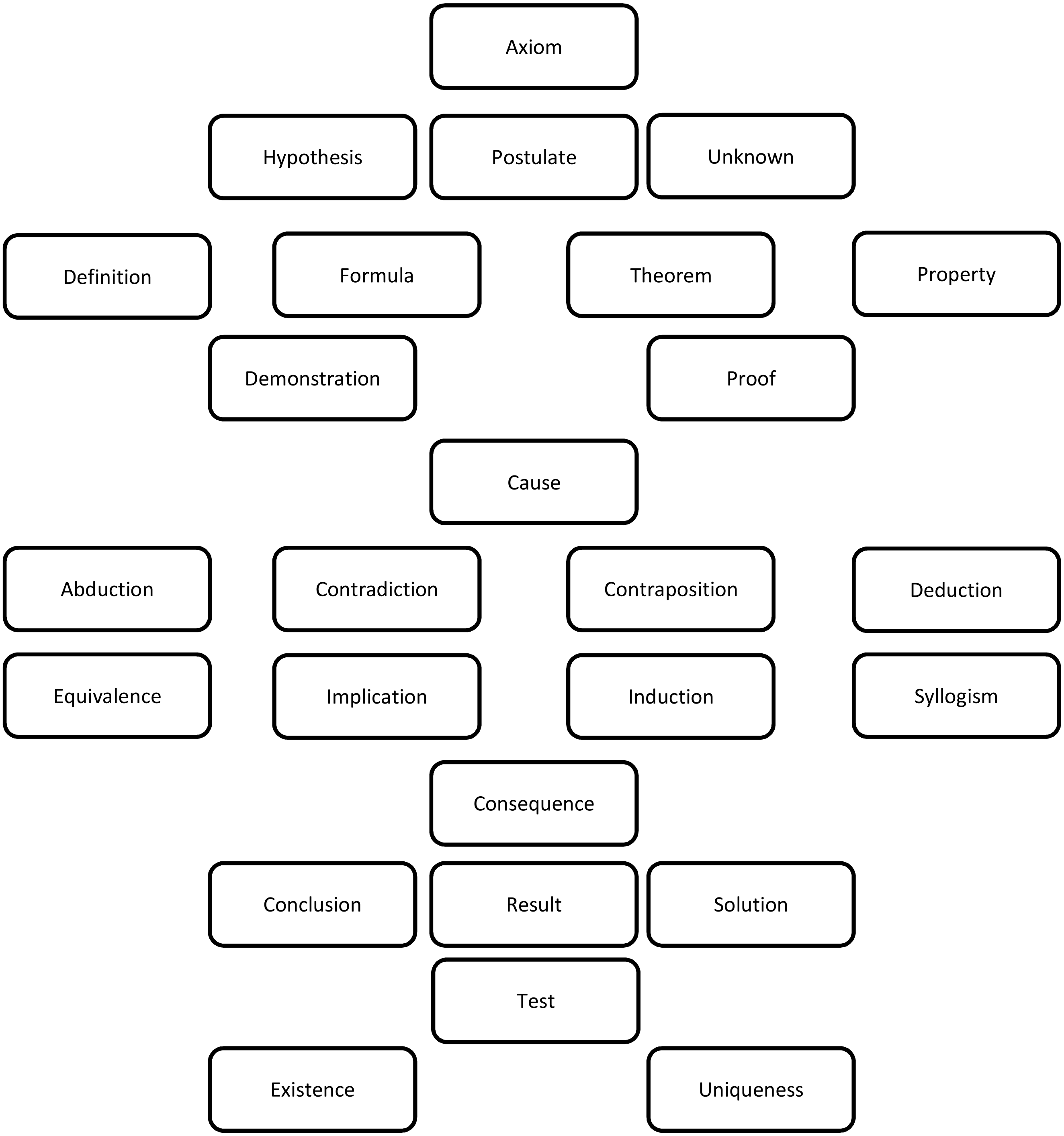
List of concepts[4] – to be augmented/refined

Concepts	**Additional information (and sub-concepts)**
Approximation	Notions of round up/down, truncation, ≈; margin of error, etc.
Algorithm	Notions of step, loop and automation.

[4] This list was developed using a variety of mathematics curriculum (see *Additional References*) as well as *The Mathematics Dictionary* by James R.C. and James G. (1992).
For related work, see the more extensive and high-level *Ontology for Engineering Mathematics* by Gruber T.R. and Olsen G.R. (1994): http://www-ksl.stanford.edu/knowledge-sharing/papers/engmath.html

Continuity	Mostly related to functions. Continuous probabilities distribution may be introduced
Dimension	1D, 2D, 3D, n dimension in algebra, etc.
Discreteness	Discretization (FFT), sequences (arithmetic and geometric), etc.
Distance	Geometric (distance, scalar product, projection, length, coordinates, etc.)
Distribution	Normal distribution (Gaussian distribution), statistical dispersion, data distribution (mean, median, variance), etc.
End and Local behavior	Maxima en minima, local and global extrema, Notion of limit
Network	Connectivity, relationships, weighing
Number	Real, complex, etc. Arabic, Roman, Indian, etc. Base 10/12/60 etc.
Operation	Unary, binary, ternary, commutative, associative. Includes notions of sum: addition, series, etc. Includes also Boolean operator in logic.
Order theory	Equality, inequality, notions of opposite and inverse, etc.
Probability	As a concept, not as a field (i.e. answer the question: what is a probability?)
Randomness	In probabilities
Rates	Growth rate, percentages, etc. Includes derivative (growth rate variation)
Representation	Includes concepts such as array or matrix; curves (hyperbola, parabola); geometric figures, etc. Includes also tree structure in programming.
Set	Finite, infinite, numbers, etc.
Simplification	In algebra (simplifying an expression such as (abc+acd/ac)
Space	Euclidian, vectorial, etc.
System	System of equations, system of parameters, system of coordinates, etc.
Transformation	Includes geometric transformations (homothecy, rotation, etc.). Introduces also the notion of *invariance*.
Variable	Introduces the notion of *function* (with specific functions such as logarithm, exponential, two variables, etc.). Can also introduce geometric equation (plan, straight line, sphere, etc.).
Vector	Includes notions of vector fields and spaces

Below we include "meta-concepts" related to **mathematical reasoning**. Those concepts include mathematical notions involved in *doing* mathematics rather than mathematics itself, which is why we separated them from the previous list.



Schema of meta-concepts

List of processes: (Source: Pr. Zal Usiskin, U. of Chicago, in private communication)

Process	Descriptions or sub-processes
Proving	Knowing general properties by name or description; justifying a result by recourse to a general property or procedure; showing how one statement leads logically to another; identifying gaps in logical arguments; deducing new results. (See meta-concepts circled in red, above)
Representing	Using a number line; displaying data with a graph of one type or another; graphing equations in the coordinate plane or in 3-space; using an algebraic description of a geometric figure; representing a network by a graph or matrix; using ordered n-tuples to describe vectors.
Modeling	Simplifying a real situation in order for it to be considered mathematically; finding mathematics (a mathematical model) that might apply to a real situation; translating results found in the mathematical model back into the situation and testing for feasibility.
Symbolizing	Measuring; mathematizing a situation; interpreting numbers, numerical expressions, and algebraic expressions with and without contexts (literacy); defining and classifying terms; estimating and dealing with estimates; scaling and dealing with scales.
Problem-solving	Recognizing a well-defined problem and the differences between what is known and what is to be found; utilizing strategies such as "draw a picture", "think backwards from what is to be found", "try a simpler problem", etc.; checking a solution using a strategy different from that used to find the solution; generalizing the problem.
Computing	Recalling basic computational facts; carrying out arithmetic and algebraic procedures and working with geometric constructions; inventing algorithms for these and other procedures; programming; using technology to carry out procedures.

Conclusion:

In conclusion, we stress that this paper is purely exploratory, and meant to stimulate conversation, not to conclude a specific list of Concepts and Processes. We hope to continue developing it with the Mathematics community in the months ahead.

Additional References

Curriculum	Link
UNESCO - Learning Metrics Task Force	Full report not published. Else, see here
Singaporean curriculum	http://www.moe.gov.sg/education/syllabuses/sciences/files/maths-primary-2013.pdf
Finnish curriculum	http://www.oph.fi/download/47672_core_curricula_basic_education_3.pdf
Massachusetts' curriculum	http://www.doe.mass.edu/frameworks/math/0311.pdf
Swiss curriculum	http://www.coe.int/t/dg4/linguistic/Source/Source2012_Sem/semSept/Linneweber_HarmoS-MatrixFr.pdf
French curriculum	http://media.education.gouv.fr/file/special_6/52/5/Programme_math_33525.pdf
Australian curriculum	http://www.acara.edu.au/verve/_resources/Australian_Curriculum_-_Maths.pdf